SELF ORGANIZATIONAL ASPECTS IN GENERAL SYSTEMS THEORY MODELS OF ORGANIZATIONS

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by

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FOREWORD

The Systems Research Center is a research and graduate study center operating in direct cooperation with all departments and divisions of Case Institute of Technology. The center brings together faculty and students in a coordinated program of research and education in the important techniques of systems theory, development, and application.

Research leading to this report was carried on by Mr. C. Fremont Sprague, Graduate Assistant, under the direction of Dr. Mihajlo D. Mesarović, Associate Professor of Engineering at Case and Director of the Adaptive and Self-Organizing Systems group of the Systems Research Center.

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Ellis A. Johnson, Director
Systems Research Center

CHAPTER 1

INTRODUCTION

1.1 The Conceptual Framework

The purpose of this investigation is to study organizations, their structure, and their development from the viewpoint of general systems theory. In general, organizational structure is characterized by

1. A division of responsibility among organizational subunits
2. The subunits' means of exercising these responsibilities
3. Measures of the subunits' performances
4. Means of signalling between subunits to indicate actions to be taken.

The responsibility which is divided is one of controlling a process so that some useful task is accomplished as the process unfolds. Item 2, then, includes the manner in which control actions are transmitted to effect the evolution of the process.

The theory of self-organizing systems is concerned with systems which are capable of modifying their own structure. This theory can be of value in operations research by supplying measures of organizational efficiency and mathematical models of changes in the structure of organi-

zations. Although reorganization has been recognized[2] as a common method of attacking organizational inefficiency, the study of structural changes in organizations, i.e. "organization theory," is mostly qualitative in character. An effective theory of self-organizing systems would contribute substantially towards quantifying organization theory.

Two different types of self-organizing systems are discussed by Mesarović,[17] causal and teleological. Causal self-organizing systems are "preprogrammed" to change their structure in response to certain stimuli. Teleological self-organizing systems, on the other hand, have a specified goal or purpose; the structure modifications are carried out by the system itself so as to pursue this goal as effectively as possible. The behavior of a teleological self-organizing system is purposeful; that is, it involves a goal-oriented choice of a particular structure from a set of possible structures. Since operations research is concerned with purposeful systems, we will confine our study to teleological self-organizing systems.

One of the difficulties encountered in the study of such systems is the absence of an ordering relation defined on the set of all possible structures of the system. Since organizations are themselves teleological self-organizing

systems, this difficulty may have contributed to the sparsity of quantification in organization theory. The main task of this thesis is to develop such an ordering relation, so that the system will have a basis for determining the "best" structure under which to operate.

The conceptual framework within which we will study teleological self-organizing systems is that of a "multi-level, multi-goal (mLnG)" system. In the mLnG representation considered in this thesis, the researcher has complete knowledge* of the components and their interactions. These components are

(1) transformation elements
(2) goal-seeking elements.

Elements of type (1) specify a set of transitions on a set of operands into a set of transforms.(5) For example, the transformation $T(a \rightarrow b,\ b \rightarrow c,\ c \rightarrow a)$, effects the transition $cab \rightarrow abc$, from the operand "cab" to the transform "abc."

Consider a set S of transformation elements, each member of S acting on the same set of operands. Elements of type (2) are assumed to have two capabilities:

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*As opposed to the "black box problem," where an experimenter attempts to deduce the contents of some unknown system by adjustment and measurement of its inputs and outputs, respectively.

(i) selection of a particular member of S

(ii) "influencing" another goal-seeking element in its choice of a particular member of S.*

Also, goal-seeking elements have a purpose attributed to them. The degree of success of achievement of this purpose depends on the transforms of the operands resulting from the transformation selected. The reader is reminded that we are not concerned with detecting the purpose; the assumption will be made that a well-defined purpose exists for each goal-seeking element in the system.

1.2 Scope and Outline of the Investigation

In the sequel, we will be concerned with the situation where the responsibility for choosing a transformation from the set S is divided among more than one goal-

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*As an example, let S consist of the two transformations $T_1(a \rightarrow b,\ b \rightarrow c,\ c \rightarrow a)$ and $T_2(a \rightarrow c,\ b \rightarrow b,\ c \rightarrow a)$, and the system be composed of S and a goal-seeking element G_1. If the purpose of the goal-seeking element G_1 is to achieve the result "acb" from the operand "cab," clearly T_2 is the transformation it should choose. If we introduce another goal-seeking element G_2 into the system, having the capability to alter G_1's goals and with the purpose of achieving the result "abaca" from the operand "cacbc," then G_2 should change G_1's desired result to "abc" in order to cause G_1 to select T_1.

seeking element with the capability of type (i) noted above. The details of how this is contrived are deferred until the next chapter; there, we will construct our system so that the contribution which each goal-seeking element makes towards selection of a particular member of **S** affects the degree of achievement of the goals of all the elements responsible for the choice. In addition, we will be interested in hierarchical arrangements of goal-seeking elements, where the group of goal-seeking elements with the divided responsibility described above are influenced by goal-seeking elements having the capability (ii) noted earlier.

The popular example of the thermostat-furnace system serves well to illustrate the concept of a multi-level, multi-goal system, as well as how the goals of the individual elements will be assumed to affect each other. We imagine a large apartment house, where each individual apartment has its own thermostat-controlled furnace. The apartment house is single-storied and the walls between apartments are quite thin, so that the thermal diffusivity between apartments is high enough for the temperature in each apartment to be affected by the temperatures in adjacent apartments. Each apartment dweller knows none of his neighbors, so that these heat sources or sinks are lumped with the environmental dis-

turbances, i.e. outside air temperature; thus, there are no abnormal behavioral consequences present, such as an apartment dweller being willing to undergo extreme discomfort in order to inflict an unusual disturbance on his neighbor.

In each apartment, the thermostat can be thought of as choosing an element from the set of transformations composed of the two elements,

1. cold air $\longrightarrow$ cold air,

2. cold air $\longrightarrow$ hot air,

in order to achieve its goal of maintaining a level of temperature in a room. The set of transformations from the over-all viewpoint, that is, for the system composed of all the thermostats and furnaces in the building, is

$$T_i \left\{ \begin{array}{l} \text{cold air} \longrightarrow Y_1 \text{ by thermostat No. 1} \\ \text{cold air} \longrightarrow Y_2 \text{ by thermostat No. 2} \\ \quad\quad\vdots \\ \text{cold air} \longrightarrow Y_N \text{ by thermostat No. N} \end{array} \right\} \; i = 1,2,3,\ldots,2^N,$$

where the Y_j's assume all combinations of the values "cold air" or "hot air" as the subscript i ranges over its indicated values. This is an example of 1LNG control.

The occupants of these apartments control their individual furnaces from the second level, by setting the goals of the thermostats. The system composed of the furnaces,

thermostats, and apartment dwellers is a 2L2NG system. If the apartment house has a "house physician," who prescribes a different room temperature to each tenant (the goal of the house physician being to keep the tenants as healthy as possible), inclusion of this individual defines a 3L(2N+1)G system. The physician indirectly exerts control on all the furnaces down through the tenant-thermostat hierarchy.

The idea of a "state of equilibrium" will be needed in the sequel. The systems which we will study in this and succeeding chapters will be dynamic systems, characterized at a particular time t by the values of a finite set of numerical quantities $x_1(t), x_2(t), \ldots, x_s(t)$. These quantities, called "state variables," constitute the components of a vector $\underline{x}(t)$, the state vector.* The "line of behavior" of a dynamic system is a trajectory in s-dimensional space, governed by differential equations if changes occur continuously in time, or by difference equations if changes occur at discrete times, $t_o, t_o + \Delta t, \ldots, t_o + n\Delta t, \ldots$. In the continuous case, if the state vector remains constant over a non-zero interval of time, however small, that state is a state of equilibirum. In the discrete case, a state of equilibrium is characterized

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*An underlined quantity denotes a vector throughout this thesis.

by the equation $\underline{x}(t_1+\Delta t) = \underline{x}(t_1)$. In either case, a state of equilibrium demonstrates the property of being unchanging in time. This investigation will be confined to the discrete case.

Notice that, because of the way we look at the system, i.e. microscopically, when the entire system is in a state of equilibrium, each element composing the system is also in a state of equilibrium. That is, each part of the system is in a state of equilibrium in the conditions provided by the other parts. One can also demonstrate the converse statement as Ashby(5) does, in order to arrive at the result: the whole system is at a state of equilibrium if and only if each part is at a state of equilibrium in the conditions provided by the other parts.

Control of a discrete dynamic system is a matter of selecting a transformation element from the set of such elements, i.e. exerting a "control action," at each transition so that the behavior is in some sense best. In the mLnG approach considered here, a performance measure is associated with each goal-seeking element.

The mLnG systems considered in this investigation have a single highest-level or "apex" goal-seeking element;

its goal is considered to be the over-all system goal. This element will always have the capability of type (ii).* We will assume the apex unit has all the information to determine the over-all optimal control law, but because of its capabilities, must influence other goal-seeking elements to implement it.

The position of a structure in the ranking of a structure set under the ordering relation (mentioned in the previous section) to be developed is determined by the efficiency of the lower level units in synthesizing the optimal control law when operating in that structure. In order to illustrate this, assume that in a mLnG system;

1. The lower level goal-seeking elements are collectively exerting control by engaging in a temporal "action-counteraction"[3] type of interplay in an attempt to arrive at a state of equilibrium, where the action of each unit is best under the conditions imposed by the other units.

2. The goal-seeking element at the apex has the ability to influence** these lower-level elements so that their collective equilibrium control actions coincide with the optimal control rule from the over-all viewpoint.

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*mLnG systems with this characteristic have been termed "indirect intervention" systems by Mesarović.[17,18]

**In order to maintain autonomy of the lower-level goal-seeking elements, this will not be in the form of a "directive" as to what control action to apply.

3. The rate at which the interplay described in (1) above approaches equilibrium is at least partially determined by the structure under which the lower-level goal-seeking elements operate.

As long as the interplay described in (1) is in the "transient phase," because of (2), the over-all optimal control law is not being achieved. Because of (3), we can rank the structures according to how well this control law is approximated; a comparison of two structures would lead to a designation of the one with the higher rate of convergence as the better. For, this would assure arrival at equilibrium, and hence optimal control, in the smaller amount of time.

Suppose we allow the highest level goal-seeking element in a mLnG system the additional (to the type (ii) capability already assumed) capability to change the existing structural arrangement of the lower level elements. With this capability and the method of ranking structures described above, this system displays behavior which we would classify as "self-organizing." The goal-seeking element at the highest level can alter the structure so as to "select" a structure from a set of different possible structures "below" it. This behavior is also purposeful, in that the criteria of choice assures the best approximation to the optimal control law, i.e. is "goal-oriented"; thus, under the assumptions made in

the previous paragraph, we would have a teleological self-organizing system.

In the next chapter we will formulate a general mathematical framework within which we shall quantify some of the ideas discussed above. Chapter II also gives definitions of the concepts which will be employed in this thesis.

Chapters III through VI develop the ideas discussed above for an important* special case, a linear system with a quadratic loss function. In Chapter III the optimal control law is derived. Chapter IV is concerned with the action-counteraction interplay between two goal-seeking elements and Chapter V discusses how a higher-level unit uses its influence to effect optimal control out of this interplay. Chapter VI is concerned with developing a self-organizing system in the manner indicated in some earlier remarks of this chapter.

Chapter VII gives the summary and conclusions.

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*since it lends itself easily to analytic treatment

CHAPTER II

MATHEMATICAL REPRESENTATION AND DEFINITION

2.1 Introduction

In this chapter a mathematical representation of an abstract multi-level, multi-goal (mLnG) system is constructed. As we mentioned in Chapter I, these systems are composed of two types of elements, transformation elements and goal-seeking elements; thus, mathematical attributes and relationships will be imputed to these two types of elements.

The mathematical arguments throughout this and later chapters will be treated so that the rigorous aspects, such as existence of solutions, interchange of limits, etc., will be omitted.

This chapter also serves to define some of the concepts and specify some of the notation utilized in later chapters.

2.2 Notational Conventions

In this and succeeding chapters, finite sequences of column vectors will be represented by the upper case letter corresponding to the lower case designation of the elements of the sequence; thus, M denotes $\underline{m}(1)$, $\underline{m}(2)$, ..., $\underline{m}(T)$, where T is finite. Subscripts common to the elements of

such a sequence will be so indicated by appending them to the upper case designation; thus X_i denotes $\underline{x}_i(1)$, $\underline{x}_i(2)$, ..., $\underline{x}_i(T)$.

We will have occasion to form column vectors by "stacking" the elements of finite sequences of vectors. This is denoted by

$$\underline{z} = [\underline{z}(1) \quad \underline{z}(2) \ \ldots \ \underline{z}(T)]'.$$

The symbol $'$ denotes transposition of vectors and matrices.

Note that $\underline{z}$ above determines a point in sT-dimensional euclidean space. This correspondence between finite sequences of vectors and points in multi-dimensional space will be utilized often in the sequel.

2.3 Dynamical Representation and Control

The core of the mathematical representation consists of a scheme that explains the trajectory or temporal behavior of the point determined by the state vector of the system, $\underline{x}(t) = [x_1(t), x_2(t), \ldots x_s(t)]'$, in s-dimensional euclidean space,

$$\Delta\underline{x}(t) = \underline{f}[\underline{x}(t), \underline{m}(t), \underline{z}(t)] \ , \ \underline{x}(0) = \underline{c}, \qquad (2.1)$$

where "Δ" is the forward difference operator in the discrete case and the time derivative in the continuous case. The vector functions $\underline{m}(t)$ and $\underline{z}(t)$ in (2.1) represent the control-

led and uncontrolled "inputs," respectively, the latter henceforth being referred to as the "disturbance." The discussion will proceed under the assumption that the process governed by (2.1) is discrete,* so that (2.1) is a difference equation, and can be rewritten as

$$\underline{x}(t+1) = \underline{S}\left[\underline{x}(t),\ \underline{m}(t+1),\ \underline{z}(t+1)\right],\ \underline{x}(0) = \underline{c}, \qquad (2.2)$$

$$t = 0, 1, 2, \ldots, T-1,$$

where T will be assumed to be finite. Equation (2.2) will subsequently be referred to as the "causal subsystem," the letter S being used to remind us that it is regarded as a system.**

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*An analogous discussion exists for the continuous case.

**In some control problems, the controlled input (to S) vector $m(t)$ is itself an output of another system: $\underline{m}(t+1) = \underline{A}[\underline{m}(t), \underline{v}(t+1)]$, where $\underline{v}$ is the controlled input to $\underline{A}$. The vector $\underline{v}$ may have only a single element, just as $\underline{m}$ could have in (2.2). This system can be integrated with the causal subsystem S by defining a new state vector $\underline{y}(t) = \underline{x}(t), \underline{m}(t)'$. Then, the vector functions $\underline{A}$ and $\underline{S}$ can be combined into a new vector function $\underline{T}$ as follows:

$$\underline{x}(t+1) = \underline{S}\ \underline{x}(t),\ \underline{A}\left[\underline{m}(t),\ \underline{v}(t+1)\right],\ \underline{z}(t+1)\}$$

$$\underline{m}(t+1) = \underline{A}\left[\underline{m}(t), \underline{\sigma}(t+1)\right]$$

becomes

$$\underline{y}(t+1) = \underline{T}\left[\underline{y}(t),\ \underline{v}(t+1),\ \underline{z}(t+1)\right],$$

which is exactly the same form as (2.2); thus, the introduction of A above causes no change in the conceptual framework at this level of generality.

In Chapter I we distinguished between goal-seeking elements which had the capabilities to select a particular transformation element from a set of such elements and goal-seeking elements which influenced other goal-seeking elements. In this chapter, the first capability, which was termed type (i), takes the form of selecting certain controlled input vectors at each transition, by regarding the process determined by (2.2) as the successive application of elements in a sequence of transformations on the initial state $\underline{x}(0) = \underline{c}$. The duality between the sequence of controlled input vectors M and the sequence of transformations

$$S[\underline{x}(0), \underline{m}(1), \underline{z}(1)], \; S[S[\underline{x}(0), \underline{m}(1), \underline{z}(1)], \underline{m}(2), \underline{z}(2)], \; \ldots \; S[S[S \; \ldots \; S[\underline{x}(0), \underline{m}(1), \underline{z}(1)], \underline{m}(2), \underline{z}(2)], \ldots,], \underline{m}(T), \underline{z}(T)]$$

is well known.(6)

Consider a goal-seeking element G having the capability of controlling the causal subsystem (2.2) through selection and implementation of the controlled-input vector sequence M. Suppose G's goal is to minimize a "loss function" $g(M, X, U)$, where $\underline{u}(t)$ is the vector of parameters in the loss function at the transition $\underline{x}(t-1) \rightarrow \underline{x}(t)$. These parameters are regarded as uncontrolled by G. The "control problem" facing G, then, is "determine M so as to minimize $g(M, X, U)$ subject to (2.2)."

The solution* of this problem is an expression for the "optimal policy"(6) or "operation"(2) M_o in terms of the sequences U and Z. This is denoted by**

$$(M)_o = M(U, Z) \tag{2.3}$$

It is clear that G must possess complete information about U and Z if it is to evaluate M_o correctly, i.e. determine the operation to be performed.

2.4 Organizational Structure and Equilibrium State

Any collection of goal-seeking elements, each of which participates in the control of a single causal subsystem, will henceforth be referred to as a "controller." Consider an organizational structure of a controller consisting of two first level goal-seeking elements G_{11} and G_{12} having the capability of type (i) and a single second level goal-seeking element G_2 having the capability of type (ii), which is characterized by the following:

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*In the mLnG theory, goal-seeking elements are assumed to have at least the problem solving capabilities of the researcher.

**Included in (2.3) and the comments following it is the case where the solution determines an optimal "feedback controller," i.e. a rule of determining M_o element-by-element as the process evolves, according to the values assumed by the state variables, denoted symbolically by

$$\underline{m}(t+1) = \underline{h}\left[\underline{x}(t),\ \underline{z}(t+1),\ \underline{u}(t+1)\right].$$

1. A reticulation (splitting up) of the causal subsystem, into two causal subsystems S_1 and S_2, by partitioning the vector function $\underline{S}$ and hence the state vector $\underline{X}$ so that

$$\underline{x}_1(t+1) = \underline{S}_1[\underline{x}(t), \underline{m}(t+1), \underline{z}(t+1)] \tag{2.4}$$

$$\underline{x}_2(t+1) = \underline{S}_2[\underline{x}(t), \underline{m}(t+1), \underline{z}(t+1)] \tag{2.5}$$

$$\underline{x}(t) = [\underline{x}_1(t)\ \underline{x}_2(t)]\ . \tag{2.6}$$

2. The causal subsystem S_i is placed under the cognizance of G_{1i}, $i = 1, 2$, i.e. G_{1i} is assumed to be aware of the functional form of $\underline{S}_i$ but not of $\underline{S}_j$, for $i \neq j$.

3. The controlled input vector is partitioned

$$\underline{m}(t) = [\underline{m}_1(t), \underline{m}_2(t)] \tag{2.7}$$

and G_{1i} is given the capability of selecting $\underline{m}_i(t)$ at each transition.

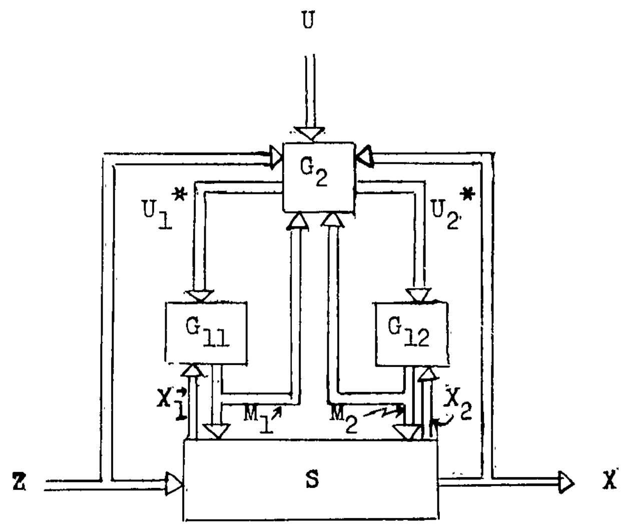
4. It is assumed that G_{1i}'s goal is to minimize the loss function $g_{1i}(M_i, X_i, U_i^*)$.

5. The second level unit G_2 has the capability to adjust the parameters U_1^* and U_2^* in the first level units' loss function. G_2 is aware of the entire system, i.e. the equation (2.2), the method of partitioning $\underline{S}$, and the goals and means of control of G_{11} and G_{12}.

6. It is assumed that G_2's goal is to minimize the loss function $g_2(M, X, U)$.

Figure 2.1 is a schematic diagram of the effects exerted on and within the 2L3G system considered here.

Figure 2.1



The dynamics of the two subsystems S_1 and S_2 are such that, under identical inputs and disturbances, the transition from $\underline{x}(t)$ to $\underline{x}(t+1)$ governed by equations (2.4), (2.5), and (2.6) is identical to the transition governed by (2.2). Equations (2.4) and (2.5) can now be rewritten as

$$\underline{x}_i(t+1) = \underline{S}_i\left[\underline{x}_i(t),\ \underline{m}_i(t+1),\ \underline{w}_i(t+1)\right] \tag{2.8}$$

for i = 1, 2, where $\underline{w}_i(t+1)$ is the vector of uncontrolled inouts to S_i and contains the effects of $\underline{m}_j(t)$ and $\underline{x}_j(t)$ for $i \neq j$. The organizational structure under consideration here can now be characterized as the simultaneous consideration of the three control oroblems

(i) of G_{1i} for $i = 1, 2$; "choose M_i so as to minimize $g_{1i}(M_i, X_i, U_i^*)$ subject to (2.8)."

(ii) of G_2; "choose U_1^* and U_2^* so that G_{11} and G_{12} will together determine M so as to minimize $g_2(M, X, U)$ subject to (2.2)."

The solution to G_{1i}'s problem is an expression analogous to (2.3),

$$(M_i)_o = M_i(U_i^*, W_i) \tag{2.9}$$

The second level element G_2 determines the optimal control law $(M)_o$ from (2.3). Denote that part of $(M)_o$ corresponding to the controlled input variables under the control of G_{1i} by $(M^i)_o$. In order for the optimal control law to be synthesized by the 2L3G controller, we must have

$$(M^i)_o = (M_i)_o, \tag{2.10}$$

so, substituting $(M^i)_o$ into the left side of (2.9) and solving for U_i^*, we get

$$(U_i^*)_o = U_i^*[(M^i)_o, W_i] \tag{2.11}$$

as the solution of G_2's control problem.

Under the conditions stated above, G_{1i} cannot, in general, determine $(M_i)_o$ from (2.9), for W_i is a function of X_j for $i \neq j$. The line of behavior X_j is affected by X_i, so that a functional dependence of W_i on X_i is established. Since X_i cannot be determined until M_i is, we have arrived

at an impasse. In order to bypass this difficulty, we propose the iterative process described below.

The finite T-stage process governed by (2.2) is imagined to be repeated an indefinite number of times. This can either be interpreted as a "real" infinite-stage process with

$$\underline{x}(0) = \underline{c} = \underline{x}(T+1) = \underline{x}(2T+2) = \dots = \underline{x}(nT+n) = \dots$$

$$\underline{z}(t) = \underline{z}(t+nT+n), \ \underline{u}(t) = \underline{u}(t+nT+n),$$

for $t = 1, 2, \dots, T$ and $n = 0, 1, 2, \dots$, or as a "fictitious" repetitive simulation of the finite T-stage process. Suppose G_{1i} adopts the following procedure in order to resolve the unknown effect of W_i. For the initial iteration of the process, i.e. from $t = 0$ to $t = T$, G_{1i} uses $\underline{w}_i(t) = \underline{0}$ for $t = 1, 2, \dots, T$ in determining its control input vector sequence from (2.9). This is denoted by $(M_i)_1$, the subscript "1" outside of the parenthesis determining the period over which this sequence runs. This notation will be extended to the other quantities of interest here; thus, $(X_i)_n$ means the sequence of $\underline{x}_i(t)$'s during the $n^{\underline{th}}$ evolution of the process. If we solve (2.8) for $\underline{w}_i(t+1)$, we can write the discrepancy in terms of the observed quantities

$$\underline{w}_i(t+1) = \underline{Q}_i\left[\underline{x}_i(t+1), \ \underline{x}_i(t), \ \underline{m}_i(t+1)\right], \qquad (2.12)$$

$$t = 0, 1, \dots, T-1.$$

Now, after G_{1i} determines $(M_i)_1$, as indicated above, we can imagine that "implementation" of this is instituted by G_{1i} actually "feeding in" to S, as its share of the inputs, the vectors $\underline{m}_i(1)$, $\underline{m}_i(2)$, ..., $\underline{m}_i(T)$ at the appropriate times. G_{1i} can, in turn, observe the subvector $\underline{x}_i(t)$ of the state vector $\underline{x}(t)$ at each time step, and with this information, can compute a $\underline{w}_i(t)$ for each integral t using (2.12). Denote the sequence thus obtained by $(W_i)_1$, and assume that G_{1i} now uses $(W_i)_1$ as the W_i in (2.9) to compute $(M_i)_2$. Then, the entire procedure can be repeated. In general, we can idealize G_{1i} as carrying on an iterative procedure, where it computes $(M_i)_n$ by the formula, analogous to (2.9),

$$(M_i)_n = M_i\left[U_i^*,\ (W_i)_{n-1}\right], \tag{2.13}$$

then uses the observations it makes on $(X_i)_n$ to compute the vector sequence according to*

$$\underline{w}_i^{\,n}(t) = \mathcal{Q}_i\left[\underline{x}_i^{\,n}(t),\ \underline{x}_i^{\,n}(t-1),\ \underline{m}_i^{\,n}(t)\right], \qquad t = 1, 2, \ldots, T, \tag{2.14}$$

which is obtained from (2.12). If the process ever yields the equality $(W_i)_n = (W_i)_{n+1}$, we would say that the process has arrived at a state of equilibrium, i.e. a state which is unchanging for subsequent periods of evolution.

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*A superscript will be used to indicate vectors of a specific sequence, so that $w_i^{\,n}(t)$ denotes a vector in the sequence $(W_i)_n$.

Notice that if a state of equilibrium is attained, i.e. $(W_i)_n = (W_i)_{n+1}$, then $(M_i)_n = (M_i)_{n+1}$ and $(X_i)_n = (X_i)_{n+1}$. Furthermore, these conditions also hold if i is replaced by j, i.e. G_{11} and G_{12} observe the event "arrival at equilibrium" simultaneously. This must hold, because the "residual" effect felt by G_{1i}, that is, the difference between the observed and the anticipated values of $\underline{w}_i(t)$, is due to the acts performed by G_{1j}; all other effects are identical over each iteration.

Unfortunately, we cannot guarantee that every such process of the type described above will arrive at a state of equilibrium. We can, however, establish sufficient conditions for this. The argument involves a technique of mathematical analysis known as the "method of successive approximations" and depends heavily on the notion of a "contraction mapping," which we now consider.

If it is possible to find a transformation $p_2 = f(p_1)$ of finite-dimensional euclidean space* into itself such that the transforms of two points in the space are nearer to one another than the original points were, f is called a "contraction mapping."[14] If we denote the distance

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*The argument generalizes to complete metric spaces.

between two points p_1 and p_2 in multi-dimensional Euclidean space by $d(p_1, p_2)$, this can be expressed in mathematical terms by requiring the existence of a positive scalar $a < 1$ such that

$$d\left[f(p_1), f(p_2)\right] \leq a\, d(p_1, p_2). \tag{2.15}$$

It is easy to see that successive applications of a contraction mapping, i.e.

$$p_2 = f(p_1),\ p_3 = f(p_2),\ \ldots,\ p_{n+1} = f(p_n),\ \ldots \tag{2.16}$$

would result in

$$p = f(p) \tag{2.17}$$

to any accuracy desired. For

$$d\left[p_{n+1}, p_n\right] = d\left[f(p_n), f(p_{n-1})\right] \leq a^{n-1}\, d\left[p_1, p_2\right]. \tag{2.18}$$

If f is a contraction mapping applied successively as indicated above,

$$\lim_{n \to \infty} d\left[p_{n+1}, p_n\right] = 0,$$

and $\lim_{n \to \infty} p_n = p$, where p is the unique solution of (2.17).

Suppose the vector of uncontrollable variables $\underline{w}_i(t)$ has q_i elements. Then, as we noted in section 2.2, the vector-sequence $\mathbf{W}_i$ determines a point in q_iT-dimensional euclidean space. The sequence of vectors $(\underline{\mathbf{W}}_i)_1, (\underline{\mathbf{W}}_i)_2, \ldots$ thus determines a sequence of points in euclidean space of q_iT dimensions. Furthermore, a mapping between successive

elements of the sequence of $\underline{W}_i$'s is implicit in equations (2.13) and (2.14); for, from (2.13), $(M_i)_n$ depends on $(W_i)_{n-1}$ and $(W_i)_n$ depends, in turn, on $(M_i)_n$. Hence, we can combine (2.13) and (2.14) to find the mapping

$$(\underline{W}_i)_{n+1} = \underline{f}\left[(\underline{W}_i)_n\right] \tag{2.19}$$

If it can be established that (2.19) is a contraction mapping, i.e. that

$$\begin{aligned} d\left\{(\underline{W}_i)_n, (\underline{W}_i)_{n+1}\right\} &= d\left\{\underline{f}\left[(\underline{W}_i)_n\right], \underline{f}\left[(\underline{W}_i)_{n-1}\right]\right\} \\ &\leqq a\, d\left\{(\underline{W}_i)_{n-1}, (\underline{W}_i)_n\right\} \end{aligned} \tag{2.20}$$

where a is a constant less than unity, our sufficient condition for convergence to equilibrium would be established.

Notice that we could have just as well investigated the mappings between $(M_i)_n$ and $(M_i)_{n+1}$ or between $(X_i)_n$ and $(X_i)_{n+1}$ for the contraction property. Furthermore, if we could establish that a mapping between equally spaced elements of any of these sequences, say between $(X_i)_n$ and $(X_i)_{n+k}$, is a contraction mapping, this would also be a sufficient condition for convergence to equilibrium. This can be easily seen by noting that the sequences

$$(X_i)_1, (X_i)_{k+1}, (X_i)_{2k+1}, \cdots$$

$$(X_i)_2, (X_i)_{k+2}, (X_i)_{2k+2}, \cdots$$

$$(X_i)_k, (X_i)_{2k}, (X_i)_{3k}, \cdots$$

all converge to the same limit, since as we stated above, the solution to (2.17) is unique; that is, it does not depend on the point from which the sequence of successive mappings begins.

A slight conceptual difficulty arises in cases where the iterative process converges, i.e. $a < 1$ in (2.20), but an infinite number of iterations is required to obtain $(W_i)_n = (W_i)_{n+1}$. Henceforth, we will refer to "achievement of equilibrium to any prescribed degree of accuracy." Given a $\delta > 0$, if the sequence $(W_i)_1$, $(W_i)_2$, ... converges, there exists a finite N such that $d\left[(W_i)_n, (W_i)_{n+1}\right] < \delta$ for $n \geq N$; thus, in this case, "achievement of equilibrium within δ" would occur at the $N^{\underline{th}}$ iteration.

The iterative process described above is conducted at the first level. Suppose $a < 1$ in (2.20) and let $(\underline{W}_i)_e$ denote the solution of $\underline{W} = \underline{f}(\underline{W})$, where $\underline{f}$ is the mapping (2.19). The "control action in equilibrium," $(M_i)_e$, and $(W_i)_e$ are connected by the relationship, analogous to (2.9),

$$(M_i)_e = M_i\left[U_i^*, (W_i)_e\right]; \tag{2.21}$$

$(M_1)_e$ has the property that it is an optimal policy for G_{11}, in the conditions U_1^* imposed by G_2, when G_{12} uses the policy $(M_2)_e$, and vice versa.

Returning to the consideration of G_2's control problem, it is seen that (2.11) contains W_i just as (2.9) does. Since G_2 is aware of the entire system and the optimal control law $(M)_o$ it can obtain $(W_1)_o$ and $(W_2)_o$ by solving (2.12) for i = 1, 2, and for t = 1, 2, ..., T when the optimal control law is applied. The following expression,

$$(U_i^*)_o = U_i^*\left[(M^i)_o,\ (W_i)_o\right], \tag{2.22}$$

obtained from (2.11), will assure that the first level iterative procedure converges to the optimal control law, i.e. that $(M^i)_o = (M_i)_e$.

If it is difficult to obtain $(W_i)_o$ by solving (2.12) as indicated above, G_2 may "assume" an equilibrium value for W_i, say $(W_i)^1$; then, using $(W_i)^1$ in (2.11), obtain $(U_i^*)^1$. The iterative procedure between the first-level elements, assuming it converges, arrives* at the equilibrium value $(W_i)_e^{\,1}$, under the conditions imposed by $(U_1^*)^1$ and $(U_2^*)^1$. G_2 now designates $(W_i)^2 = (W_i)_o^{\,1}$ and repeats the process for $(W_i)^2$ to determine a new first-level value $(W_i)_e^{\,2}$. Continuation of this process defines a "nested" iterative procedure, where each single step in the $(W_i)^n$ iteration involves the entire $(W_i)_n$ iteration described earlier. Convergence

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*Within some prescribed degree of accuracy.

requires that

$$(w_i)^{n+1} = h\left[(w_i)^n\right] \tag{2.23}$$

be a contraction mapping, as well as f in (2.19). This case will not be considered further in this chapter. It will appear again in Chapter VII.

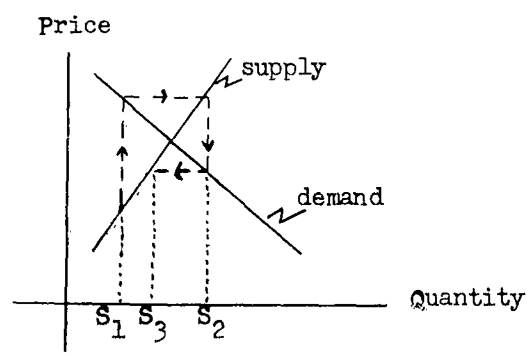
It is interesting to examine the relation between the concept of equilibrium, as we use it here, to similar ideas in other theories. In mechanics, the idea of equilibrium plays a large role, particularly in the sub-area called "statics". A weight, hanging by a chain, is in equilibrium, the downward force exerted by gravity being exactly balanced by the upward force of the chain. In "dynamics", the concept is also important, and begins to resemble our ideas above. If we imagine the weight to be pushed so that s swinging motion is introduced, the forces of friction impose a contraction mapping on this system, in that the extreme point of the arc of each swing is closer to the "point of rest", or "equilibrium point" in our notion, than the extreme point of the arc of the previous oscillation. This is an example of "Lyapunov stability."[23]

Game theory also uses the concept of "equilibrium point"[16,19] in a manner quite similar to that described

above. The iterative process described earlier can be thought of as either a temporal repetition of a "game of prescribed duration"(11) or as the "fictitious play"(16) of such a game. Game theory also utilizes the idea of "mappings with a single fixed point"(19) to determine points of equilibrium. A particular application of game theory to the types of problems arising in multi-level control is discussed in Chapter IV.

Various theories in economics utilize the idea of equilibrium to explain prices based on laws of supply and demand. The "dynamic cobweb"(21) concept is based on the fact that the present supply creates prices causing a specific demand which, in turn, leads to a new level of supply. This procedure is somewhat analogous to the iterative process described earlier in that it defines a contraction mapping under certain conditions. Notice in Figure 2.2 that the distance between S_2 and S_3 is less than the distance between S_1 and S_2.

Figure 2.2



As we stated in Chapter I, we propose the rate of convergence to equilibrium as a means of comparing the effectiveness of different structures of a mLnG system. Referring back to the characterization of a structure made at the beginning of this section, it is clear that we can represent each element of the structure set considered here by the two vectors $\underline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_s)$ and $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_r)$, called "selector vectors", having the following properties:

$\sigma_j = i$ if $x_j(t)$ is under the cognizance of G_{1i},

where $i = 1, 2,$ $j = 1, 2, \ldots, s$, for the special case above, and

$\mu_k = i$ if $m_i(t)$ is manipulated by G_{1i}

where $k = 1, 2, \ldots, r$ and again $i = 1, 2$. For example, with five state variables and 6 controllable variables

$\underline{\sigma} = (1\ 2\ 1\ 1\ 2)$

$\underline{\mu} = (2\ 2\ 1\ 2\ 1\ 1)$

would mean

$$\underline{x}_1(t) = \left[x_1(t)\ x_3(t)\ x_4(t)\right]', \quad \underline{x}_2(t) = \left[x_2(t)\ x_5(t)\right]'$$

and

$$\underline{m}_1(t) = \left[m_1(t)\ m_2(t)\ m_4(t)\right]', \quad \underline{m}_2(t) = \left[m_3(t)\ m_5(t)\ m_6(t)\right]'.$$

If the rate of convergence of the iterative process between the two first level goal-seeking elements varies as

the structure changes, then that structure which affords the most rapid rate of convergence is best. For, suppose the repetition of the finite-stage process is regarded as an infinite-stage periodic process, as mentioned earlier. The faster the rate of convergence, the better the approximation of the sequence $(M)_1$, $(M)_2$, $(M)_3$, ... to the optimal control law for this process, $(M)_o$, $(M)_o$, $(M)_o$, ... Here, $(M)_n$ denotes the combined control actions $(M_1)_n$ and $(M_2)_n$ for the $n^{\underline{th}}$ "period", and $\lim_{n \to \infty} (M)_n = (M)_o$, through G_2's "influence" as noted earlier.

2.5 Self-Organizational Activity

In this section G_2 is allowed the additional* capability of changing the organizational structure of the controller. An example of a "structure change" as it is regarded in this investigation would be "take $x_k(t)$ from G_{12}'s cognizance and place it in G_{11}'s, i.e. change σ_k from 2 to 1, making appropriate changes in the loss function, the reticulation specified by (2.4), (2.5), and (2.6), and the means of influencing G_{11} and G_{12}." The contraction factor a in (2.20) could be regarded as a structural parameter, since it decreases as the rate of convergence increases; thus, if the

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*To the capability of type ii already granted it.

above change produces a reduction in a, it would be classified as a change for the good. The variation of the parameter a, then, establishes an ordering relation over the set of structures considered here.

The specification of U_1^* and U_2^* by G_2 according to (2.22) can be regarded as a mathematical convenience. It allows us to hold the equilibrium point constant while varying the structure. This results in a concentration on the self-organizational aspects, the main objective of this thesis.

Let us regard the iterative process as determining a dynamic system whose state trajectory is $X_1, X_2, \ldots$. It is interesting to compare the problem of which structure G_2 should choose with a conventional control problem(9); "given a dynamical system

$$\underline{y}(t+1) = \underline{P}\left[\underline{y}(t),\ \underline{f}(t)\right],$$

find the vector function $\underline{f}(t)$ such that given an initial value $\underline{y}(0)$, the system reaches equilibrium in the shortest possible time." The "structure-choice problem" is strikingly similar; "given a dynamical system

$$\underline{X}_{n+1} = Q_a\left[(\underline{X})_n\right],$$

where the transformation Q_a depends on the structure and has a contraction factor a, find the structure such that given an

initial value $(\underline{X})_1$, the system reaches equilibrium in the shortest possible time."

Throughout this chapter the discussion has been held to a high degree of generality. Such things as the question of the stability of equation (2.4) under the "forcing function" represented by M and the "realizability" of the solution M have been assumed away. In the remainder of this thesis, we will investigate special cases which, although simple, will contain all of the features brought out here. In particular, Chapter VI is concerned with the application to a specific case of the general statement made above, namely that the rate at which a reticulated mLnG system approaches equilibrium is a good measure of the effectiveness of the reticulation.

CHAPTER III

OPTIMAL CONTROL FOR A 1L1G SYSTEM

3.1 Introduction

The purpose of this chapter is to describe a particular control problem which we will be concerned with in the sequel. The most important feature of this control problem is that it is solvable analytically using standard techniques. Although the problem is simple, it still contains all the relevant components of the general mathematical representation stated in the previous chapter. The solution will be derived under the assumption that the controller is of the simplest possible structure, i.e. single-level single-goal, since this is the form in which it will be applied later. Of particular interest will be the formulas expressing the operation to be performed by this single goal unit in terms of the state variables, uncontrollable variables, and disturbances; we will use these many times in what follows.

The notational convention of representing sequences of vectors by upper case letters is carried on in this chapter.

The situation we will consider here is this: a system S is such that changes in its state vector $\underline{x}(t)$ occur at discrete instances of time. These changes can be described

by the linear, dynamic, vector-matrix difference equation

$$\underline{x}(t+1) = A\underline{x}(t) + \underline{z}(t+1) + \underline{m}(t+1), \qquad (3.1)$$

over a finite number of time periods, $t = 0, 1, 2, \ldots, T-1$, with initial conditions $\underline{x}(0) = \underline{c}$. The vector $\underline{z}(t)$ is considered to be an uncontrolled input or disturbance which is predictable without error over the entire time domain, hence is essentially a vector of parameters. The vector $\underline{m}(t)$ is the controlled input vector, having the same number of elements as $\underline{x}(t)$. The matrix A is non-singular and constant over the period of interest.

A goal unit G is charged with guiding the state trajectory $\underline{x}(1), \underline{x}(2), \ldots, \underline{x}(T)$ of S along a path which minimizes

$$g(M,X) = \sum_{t=1}^{T} \left[\underline{x}(t) - \underline{u}(t)\right]' \left[\underline{x}(t) - \underline{u}(t)\right] \qquad (3.2)$$

$$+ \sum_{t=1}^{T} \underline{m}'(t)\, D\underline{m}(t),$$

where D is a positive-definite diagonal matrix and U is a known vector sequence, with the vectors $\underline{u}(t)$ having the same number of elements as $\underline{x}(t)$, and with a definite element-by-element association between the two. The seqence U can be thought of as determining an "ideal trajectory" along

which G "desires" the actual state trajectory determined by X to move. For example, if $x_i(t)$ is "steel capacity at t," $u_i(t)$ would be "desired steel capacity at t." The performance criteria $g(M,X)$, then, represents a balance between the cost of tolerating nonideal behavior and the cost of doing something about it.

The set of acts available to G at any particular transition, say at the transition

$$\underline{x}(t-1) \longrightarrow \underline{x}(t),$$

is simply the numerical adjustments $m_1(t)$, $m_2(t)$, ..., $m_s(t)$ made on the corresponding elements of the vector $A\underline{x}(t-1) + \underline{z}(t)$. The "operation" or "optimal policy" performed by G is emission of the vector-sequence of these adjustments, M, which minimizes (3.2). The operation is assumed to be realizable. We now proceed to find the operation, i.e. solution to the minimization problem.

Notice that the entire process governed by (3.1) over the times 0, 1, 2, ..., T, can be written as a "closed form" solution of the difference equation (3.1), in terms of the initial states and subsequent control input vectors and disturbances. This is obtained by applying the formula, derived in Appendix A,

$$\underline{x}(k) = A^k \underline{o} + \sum_{j=1}^{k} A^{k-j}\left[\underline{m}(j) + \underline{z}(j)\right].$$

This can be further abbreviated in vector-matrix form:

$$\underline{X} = K[\underline{M} + \underline{Z}] + \underline{C} , \tag{3.3}$$

where $K = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ A & I & 0 & \cdots & 0 \\ A^2 & A & I & \cdots & 0 \\ & & \cdots & & \\ A^{T-1} & A^{T-2} & \cdots & A & I \end{bmatrix}$, a matrix with matrix

elements, and

$$\underline{C} = \begin{bmatrix} A\underline{c} \\ A^2\underline{c} \\ \vdots \\ A^T\underline{c} \end{bmatrix} .$$

The "loss function" now becomes

$$g(\underline{M},\underline{X}) = [\underline{X} - \underline{U}]'[\underline{X} - \underline{U}] + \underline{M}'E\underline{M} ,$$

where

$$E = \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ 0 & D & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & & \vdots \\ 0 & 0 & 0 & \cdots & D \end{bmatrix} .$$

3.2 Derivation of the Optimal Control Rule

The simple nature of this particular control problem* is now evident. The problem is "minimize

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*This is simply the sT-dimensional analogue of the scalar problem, "minimize $(x-u)^2 + em^2$, subject to $x = k(m+z) + c$, where $c > 0$."

$(\underline{X} - \underline{U})'(\underline{X} - \underline{U}) + \underline{M}'E\underline{M}$, subject to $\underline{X} = K(\underline{M} + \underline{Z}) + \underline{C}$, where E is a positive-definite matrix." It can be solved either by employing the method of Lagrange's multipliers, or by substitution of $K(\underline{M} + \underline{Z}) + \underline{C}$ for $\underline{X}$ directly in g; we will do the latter. Let $g(\underline{M})$ be used to denote that this substitution has been made;

$$
\begin{aligned}
g(\underline{M}) &= \left[K(\underline{M} + \underline{Z}) + \underline{C} - \underline{U}\right]' \cdot \left[K(\underline{M} + \underline{Z}) + \underline{C} - \underline{U}\right] \qquad (3.4) \\
&\quad + \underline{M}'E\underline{M} \\
&= \underline{M}'(K'K + E)\underline{M} + 2(K\underline{Z} + \underline{C} - \underline{U})'K\underline{M} \\
&\quad + (K\underline{Z} + \underline{C} - \underline{U})'(K\underline{Z} + \underline{C} - \underline{U}).
\end{aligned}
$$

The diagonal matrix E is positive definite and, since K is non-singular, K'K is also positive definite. Hence, K'K + E is positive definite. The positive definiteness of K'K + E is a necessary and sufficient condition for g to be a strictly convex function of the elements of $\underline{M}$.

In order to find the stationary point of (3.4), we differentiate $g(\underline{M})$ with respect to the elements of $\underline{M}$ and solve the system of linear algebraic equations which results* from setting these derivatives equal to zero:

---

*The derivative of the quadratic form g with respect to the vector $\underline{M}$ is a vector of the same dimension as $\underline{M}$. The details of this representation are relegated to Appendix B.

$$\frac{\partial g}{\partial \underline{M}} = 2E\underline{M} + 2K'K\underline{M} + 2K'(K\underline{Z} + \underline{C} - \underline{U}) = \underline{0} \tag{3.5}$$

$$\Longrightarrow \underline{M} = (K'K + E)^{-1}K'(\underline{U} - \underline{C} - K\underline{Z}). \tag{3.6}$$

The minimum value of $g(\underline{M})$ can now be found by substituting $\underline{M}$ as obtained from (3.6) into $g(\underline{M})$, as given by (3.4); thus,

$$g_{\min}(\underline{M}) = (\underline{U} - \underline{C} - K\underline{Z})'\left[I - K(K'K + E)^{-1}K'\right](\underline{U} - \underline{C} - K\underline{Z}). \tag{3.7}$$

To determine the state vector sequence $\underline{X}$ which results from "implementing" the operation, (3.6) is substituted into (3.3), resulting in

$$\underline{X} = K(K'K + E)^{-1}K'\underline{U} + K\left[I - (K'K + E)^{-1}K'K\right]\underline{Z}$$
$$+ \left[I - K(K'K + E)^{-1}K'\right]\underline{C},$$

or

$$\underline{X} = K(K'K + E)^{-1}(K'\underline{U} + E\underline{Z}) + \left[I - K(K'K + E)^{-1}K'\right]\underline{C}. \tag{3.8}$$

3.3 Stability

In order to investigate the stability of the above process, it is first necessary to introduce the concept of a "norm" of both vectors and matrices. Define the norm of a vector as

$$||\underline{X}_i|| = \left[\underline{X}_i'\underline{X}_i\right]^{1/2} = \left[\sum_{t=1}^{T} \underline{x}_i'(t)\underline{x}_i(t)\right]^{1/2}. \tag{3.9}$$

The norm of a matrix A is said to be "compatible" with the norm of a vector as defined above if

$$||A\underline{x}|| \leq ||A|| \; ||\underline{x}||. \tag{3.10}$$

A method of constructing a matrix norm so as to satisfy this compatibility condition is to apply the formula

$$||A|| = \max_{||\underline{x}|| = 1} ||A\underline{x}|| \; . \tag{3.11}$$

Norms of matrices also have the properties

$$||AB|| \leq ||A|| \; ||B||, \tag{3.12}$$

$$||A+B|| \leq ||A|| + ||B||, \tag{3.13}$$

and

$$||I|| = 1. \tag{3.14}$$

In addition, Halmoz[13] and Fadeeva[10] prove that

$$||A|| = \left[\mu_1(A'A)\right]^{1/2} \tag{3.15}$$

and

$$A \text{ symmetric} \Longrightarrow ||A|| = \max\left[|\mu_1(A)|, |\mu_n(A)|\right]. \tag{3.16}$$

In particular:

$$A \text{ positive definite} \Longrightarrow ||A|| = \mu_1(A). \tag{3.17}$$

In (3.15), (3.16), and (3.17) immediately above, $\mu_i(A)$ denotes the $i^{\underline{th}}$ characteristic root of A, the convention $\mu_n \leqq \mu_{n-1} \leqq \mu_2 \leqq \mu_1$ being understood here.

The process described by (3.8) will be termed "stable" if for each $e > 0$, there exists a $\delta > 0$ such that

any trajectory $\underline{y}(1)$, $\underline{y}(2)$, ..., $\underline{y}(T)$ satisfying the condition $||\underline{x}(0) - \underline{y}(0)|| \leq \delta$ satisfies $||\underline{x}(t) - \underline{y}(t)|| < e$ for $t = 1, 2, \ldots, T$. In order to demonstrate the stability of (3.8), rewrite $\underline{C}$ as

$$\underline{C} = \begin{bmatrix} A\underline{c} \\ A^2\underline{c} \\ \vdots \\ A^T\underline{c} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & \cdots & 0 \\ 0 & A^2 & 0 & \cdots & 0 \\ 0 & 0 & A^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & A^T \end{bmatrix} \begin{bmatrix} \underline{x}(0) \\ \underline{x}(0) \\ \vdots \\ \underline{x}(0) \end{bmatrix} = J\underline{C}_x; \tag{3.18}$$

since $\underline{x}(0) = \underline{c}$. Letting

$$\underline{C}_y = \begin{bmatrix} \underline{y}(0) \\ \underline{y}(0) \\ \vdots \\ \underline{y}(0) \end{bmatrix} \tag{3.19}$$

we have, from (3.8),

$$||\underline{X}-\underline{Y}|| = \left\{ \sum_{t=1}^{T} [x(t) - y(t)]'[x(t) - y(t)] \right\}^{1/2} \tag{3.20}$$

$$= ||[I - K(K'K + E)^{-1}K']J(\underline{C}_x - \underline{C}_y)||,$$

where $\underline{X}$ and $\underline{Y}$ are both trajectories determined by (3.8) but with different initial conditions. From (3.20), and taking note of (3.10) and (3.12),

$$||\underline{x}(t) - \underline{y}(t)|| = \left\{ [\underline{x}(t) - \underline{y}(t)]'[\underline{x}(t) - \underline{y}(t)] \right\}^{1/2}$$

$$\leq ||\underline{X}-\underline{Y}|| \leq ||I - K(K'K+E)^{-1}K'||\cdot||J||\cdot||\underline{C}_x-\underline{C}_y||. \tag{3.21}$$

Now, it can be shown that $||I-K(K'K+E)^{-1}K'|| < 1$ (for an indication of how this is done, see the proof of Theorem 5.3.1 in the fifth chapter) and, from (3.11) and (3.18), it is clear that $||J||=||A||$. Also

$$||\underline{c}_x-\underline{c}_y|| = \left\{ \sum_{j=1}^{T} [\underline{x}(0)-\underline{y}(0)]'[\underline{x}(0)-\underline{y}(0)] \right\}^{1/2} \tag{3.22}$$

$$= T^{1/2}\,||\underline{x}(0)-\underline{y}(0)||.$$

In (3.22), we have utilized another property of the norm, namely that if a is a scalar,

$$||a\underline{x}|| = |a|\cdot||\underline{x}||. \tag{3.23}$$

As a result of the statements immediately above, it is apparent from (3.21) that

$$||\underline{x}(t)-\underline{y}(t)|| < ||A||T^{1/2}||\underline{x}(0)-\underline{y}(0)||;$$

thus, given an $e > 0$, letting $\delta = e/(||A||T^{1/2})$, it is easily seen that if $||\underline{x}(0)-\underline{y}(0)|| < \delta$ then $||\underline{x}(t)-\underline{y}(t)|| < e$, so that the process described by (3.8) is stable. This definition of stability is a specialization of "Liapunov stability" (see Struble[23]).

3.4 Sensitivity of the Loss Function to the Variation of Certain Parameters

It is interesting to examine the value of g min, as given by (3.7), for different values of the elements of the positive definite diagonal matrix E, which are the

elements of D (in (3.2)) repeated T times.* This can be rigorously investigated by proving

<u>Theorem 3.1</u>: An increase (decrease) in at least one element of E in (3.7) results in a decrease (increase) in g_{min} ($\underline{M}$) as given there; these manipulations are assumed to be constrained so as to maintain the positive-definiteness of E.

<u>Proof</u>: Recall that $K'K+E$, and hence $(K'K+E)^{-1}$ and $K(K'K+E)^{-1}K'$, are positive definite. Now

$$[K(K'K+E)^{-1}K']^{-1} = I + (K^{-1})'EK^{-1}. \tag{3.24}$$

Consider the quadratic form

$$Q = \underline{q}'[K(K'K+E)^{-1}K']^{-1}\underline{q} = \underline{q}'\underline{q} + \underline{p}'E\underline{p},$$

where $\underline{p} = K^{-1}\underline{q}$. Then

$$Q = \sum_{i=1}^{n} [q_i^2 + e_{ii}p_i^2], \tag{3.25}$$

since E is a diagonal matrix. Now, it is clear from (3.25) than an increase (decrease) in any e_{ii} results in an increase (decrease) in Q, for constant $\underline{p}$ and $\underline{q}$. Let T denote the

---

*The intuitive notion that g min increases (decreases) if any of the control costs increase (decrease) can be seen by regarding the matrix $I-K(K'K+E)^{-1}K'$ in (3.7) as a generalization of the positive number $1-k^2/(k^2+e)$, with $e > 0$. The right hand side of (3.7) can then be thought of as a generalization of $(u-c-kz)^2[1-k^2/(k^2+e)]$ where u, c, k, and z are scalars. Increasing e increases the expression in [], which in turn increases $(u-c-kz)^2[1-k^2/(k^2+e)]$ if $(u-c-kz)^2$ is held constant.

orthogonal transformation* which reduces $K(K'K+E)^{-1}K'$ to canonical form, and let

$$\underline{r} = T\underline{q}$$

Then

$$Q = \sum_{i=1}^{n} \frac{1}{\mu_i} r_i^2$$

so that an increase (decrease) in Q must result in a decrease (increase) in at least one of the characteristic roots μ_1, μ_2, ..., μ_n of $[K(K'K+E)^{-1}K']$. If we let $\underline{s} = T(K\underline{Z}+\underline{C}-\underline{U})$ then (3.7) becomes

$$g_{min}(\underline{M},\underline{X}) = (K\underline{Z}+\underline{C}-\underline{U})'(K\underline{Z}+\underline{C}-\underline{U}) - \sum_{i=1}^{n} \mu_i s_i^2 ,$$

so that a decrease(increase) in any of the μ_i's clearly causes an increase (decrease) in g_{min} $(\underline{M},\underline{X})$, Q.E.D.

3.5 Dynamic Programming Solution

A method of obtaining the minimum of (3.2) subject to (3.1) by dynamic programming is presented in Appendix C. This should be a preferable method of solution when the number of transitions of the process, T, becomes very large. The dynamic programming technique substitutes solving T systems of equations, each s x s, for solving one sT x sT system of equations as was done above, where s is the number of elements in $\underline{x}(t)$ and $\underline{m}(t)$.

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*T exists since $K(K'K+E)^{-1}K'$ is positive-definite.

CHAPTER IV

SOME ASPECTS OF A 1L2G CONTROL PROBLEM

4.1 Introduction

In this chapter we investigate the decomposition of the control problem of Chapter III into two similar smaller problems. Two goal-seeking elements G_{11} and G_{12}* will each be concerned with one of these smaller problems. All the notational designations of Chapter III are carried over to this chapter. In addition, it is understood in what follows that $i = 1, 2,\ \ j = 1, 2,$ and $i \neq j$.

Consider the following partitioning of the vectors $\underline{x}(t)$, $\underline{z}(t)$, $\underline{m}(t)$, $\underline{u}(t)$, and the matrices A and D;

$$\underline{x}(t) = [\underline{x}_1(t)\ \underline{x}_2(t)]',$$

$$\underline{z}(t) = [\underline{z}_1(t)\ \underline{z}_2(t)]',$$

$$\underline{m}(t) = [\underline{m}_1(t)\ \underline{m}_2(t)]',$$

$$\underline{u}(t) = [\underline{u}_1(t)\ \underline{u}_2(t)]',$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad D = \begin{bmatrix} D_{11} & 0 \\ 0 & D_{12} \end{bmatrix},$$

such that (3.1) can be expressed as

*The two-digit subscripts "11" and "12" are used in agreement with the notation employed in other parts of this thesis.

$$\underline{x}_1(t+1) = A_{11}\underline{x}_1(t) + A_{12}\underline{x}_2(t) + \underline{z}_1(t+1) + \underline{m}_1(t+1) \qquad (4.1)$$

$$\underline{x}_2(t+1) = A_{21}\underline{x}_1(t) + A_{22}\underline{x}_2(t) + \underline{z}_2(t+1) + \underline{m}_2(t+1) \qquad (4.2)$$

and the loss function (3.2) is "separable", i.e.

$$g(M,X) = g_{11}(M_1,X_1) + g_{12}(M_2,X_2) \qquad (4.3)$$

where

$$g_{1i}(M_i,X_i) = \sum_{t=1}^{T} \left[\underline{x}_i(t)-\underline{u}_i(t)\right]'\left[\underline{x}_i(t)-\underline{u}_i(t)\right] \qquad (4.4)$$

$$+ \sum_{t=1}^{T} \underline{m}_1'(t)\, D_{1i}\underline{m}_i(t) \; .$$

The control problem of concern to G_{1i} is "choose M_i so as to minimize $g_{1i}(M_i,X_i)$ subject to (4.i)."

The difference between this control problem and the one studied in Chapter III is the presence of the "cross-coupling" disturbance $A_{ij}\underline{x}_j(t)$. As a result, the total disturbance at each transition, call it

$$\underline{w}_i(t+1) = A_{ij}\underline{x}_j(t) + \underline{z}_i(t+1), \qquad (4.5)$$

is functionally dependent on the state vector sequence* X_i. In Chapter III and Appendix C, Z, of which W_i is the analogue,

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*This is strictly a mathematical statement. It occurs by way of X_j in (4.5), hence on W_j and finally X_i by interchanging i and j in (4.5). An organizational interpretation would be that the actions M_1 taken by G_{11} affect the trajectory X_2 of G_{12}'s state variables by way of (4.1) and (4.5) and vice versa.

in independent of X and the solutions obtained there are based on this fact. Methods of bypassing this difficulty so that these solutions can be applied are considered in later sections of this chapter.

We will be concerned with finding "equilibrium" policies.* These are denoted by $(M_1)_e$ and $(M_2)_e$, and have the property that $(M_1)_e$ is the solution to G_{11}'s control problem when G_{12} uses $(M_2)_e$ and vice versa. The manner in which the concept of equilibrium is utilized in this thesis has been outlined in Chapters I and II. This chapter is a step in the application of these ideas to certain linear systems.

4.2 1L2G Control under Perfect Information

In this section we make the following

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*The 1L2G control problem formulated here is an example of a "game of prescribed duration."[11] This nomenclature refers to a multistage process in which each of n players (here, n=2) exerts a control on the position of the process. In the case studied here, the process is governed by (3.1). The control exerted by the "player" G_{1i} is through M_i and G_{1i}'s "payoff" is determined according to how well it keeps the value of $g_{1i}(M_i,X_i)$ down. The equilibrium policy is analogous to an equilibrium point in game theory,[16] in the sense that $(M_1)_e$ is optimal for G_{11} when G_{12} uses $(M_2)_e$ and vice versa.

<u>Assumption</u> 4.2.1 The 1L2G system is a "perfect information"[17,18] system, i.e. G_{11} and G_{12} both have complete knowledge of the entire system, of U and Z, and of each other's goals. Note that the statement of G_{1i}'s control problem in the previous section did not specify that G_{1i} have this information.

In what follows, equilibrium policies for G_{11} and G_{12} will be found using assumption 4.2.1. These policies will provide a benchmark with which to compare policies found by other techniques in later sections.

Recall, from Chapter III, the "unfolded" representation of the closed form solution of the difference equations (4.1) and (4.2), with notational changes appropriate to this chapter,

$$\underline{X}_i = \begin{bmatrix} \underline{x}_i(1) \\ \underline{x}_i(2) \\ \vdots \\ \underline{x}_i(T) \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ A_{ii} & I & 0 & \cdots & 0 \\ A_{ii}^2 & A_{ii} & I & \cdots & 0 \\ \vdots & & & & \vdots \\ A_{ii}^{T-1} & A_{ii}^{T-2} & A_{ii}^{T-3} & \cdots & I \end{bmatrix} \begin{bmatrix} \underline{w}_i(1) + \underline{m}_i(1) \\ \underline{w}_i(2) + \underline{m}_i(2) \\ \vdots \\ \underline{w}_i(T) + \underline{m}_i(T) \end{bmatrix}$$

$$+ \begin{bmatrix} A_{ii}\underline{c}_i \\ A_{ii}^2\underline{c}_i \\ \vdots \\ A_{ii}^T\underline{c}_i \end{bmatrix} = K_i(\underline{W}_i + \underline{M}_i) + \underline{C}_i . \qquad (4.6)$$

Equation (4.5) can be represented as

$$\underline{\mathbf{W}}_i = \begin{bmatrix} \underline{w}_i(1) \\ \underline{w}_i(2) \\ \\ \\ \underline{w}_i(T) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ A_{ij} & 0 & \cdots & 0 & 0 \\ 0 & A_{ij} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & A_{ij} & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_j(1) \\ \underline{x}_j(2) \\ \\ \\ \underline{x}_j(T) \end{bmatrix}$$

$$+ \begin{bmatrix} \underline{z}_i(1) \\ \underline{z}_i(2) \\ \vdots \\ \underline{z}_i(T) \end{bmatrix} = L_{ij}\underline{\mathbf{X}}_j + \underline{\mathbf{Z}}_i \ . \qquad (4.7)$$

The loss function (4.4) is rewritten as

$$g_{1i}(\underline{M}_i,\underline{X}_i) = (\underline{X}_i-\underline{U}_i)'(\underline{X}_i-\underline{U}_i) + \underline{M}_i'E_{1i}\underline{M}_i \qquad (4.8)$$

where

$$E_{1i} = \begin{bmatrix} D_{1i} & 0 & \cdots & 0 \\ 0 & D_{1i} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & D_{1i} \end{bmatrix};$$

in accordance with the unfolded representation.

From (3.6) in Chapter III, it is seen that the minimum of $g_{1i}(\underline{M}_i,\underline{X}_i)$ subject to (4.6) occurs for

$$\underline{M}_i = (K_i'K_i+E_{1i})^{-1}K_i'(\underline{U}_i-K_i\underline{\mathbf{W}}_i-\underline{C}_i) \qquad (4.9)$$

If we combine the system of equations (4.9) for i=1 with the system (4.9) for i=2, we get a set of sT linear algebraic equations in the sT unknown elements of $\underline{M}_1$ and $\underline{M}_2$. In order to express these in terms of $\underline{C}$, $\underline{Z}$, and $\underline{U}$ only, it is necessary

to eliminate the $\underline{X}_1$ and $\underline{X}_2$ which is implicitly contained in $\underline{W}_2$ and $\underline{W}_1$, respectively. Also, it will be helpful to rewrite the system of equations (4.9) as

$$(K_i')^{-1}(K_i'K_i+E_{1i})\underline{M}_i = \underline{U}_i - K_i\underline{W}_i - \underline{C}_i. \qquad (4.10)$$

Substituting (4.7) into (4.6), it follows that

$$\underline{X}_i = K_iL_{ij}\underline{X}_j + K_i\underline{Z}_i + K_i\underline{M}_i + \underline{C}_i. \qquad (4.11)$$

Now, (4.6) with i replaced by j, substituted into (4.11) yields

$$\underline{X}_i = K_i\underline{M}_i + K_iL_{ij}\left[K_j(\underline{W}_j+\underline{M}_j) + \underline{C}_j\right] + K_i\underline{Z}_i + \underline{C}_j \qquad (4.12)$$

which, from (4.7) with i and j interchanged, is

$$\underline{X}_i = K_i\underline{M}_i + K_iL_{ij}K_jL_{ji}\underline{X}_i + K_iL_{ij}K_j\underline{Z}_j$$
$$+ K_iL_{ij}K_j\underline{M}_j + K_iL_{ij}\underline{C}_j + \underline{C}_i + K_i\underline{Z}_i \qquad (4.13)$$

or

$$\underline{X}_i = (I-K_iL_{ij}K_jL_{ji})^{-1}(K_i\underline{M}_i+K_iL_{ij}K_j\underline{Z}_j+K_iL_{ij}K_j\underline{M}_j$$
$$+ K_iL_{ij}\underline{C}_j+K_i\underline{Z}_i+\underline{C}_i). \qquad (4.14)$$

Equation (4.14) is an expression of $\underline{X}_i$ in terms of $\underline{M}_i$ and $\underline{M}_j$; hence, using (4.14) with i and j interchanged, (4.10), and (4.7), the following result is obtained;

$$(K_i')^{-1}(K_i'K_i+E_{1i})\underline{M}_i = \underline{U}_i - K_i(L_{ij}\underline{X}_j+\underline{Z}_i) - \underline{C}_i$$
$$= \underline{U}_i - K_iL_{ij}(I-K_jL_{ji}K_iL_{ij})^{-1}(K_j\underline{M}_j+K_jL_{ji}K_i\underline{Z}_i$$
$$+ K_jL_{ji}K_i\underline{M}_i+K_jL_{ji}\underline{C}_i+K_j\underline{Z}_j+\underline{C}_j) - K_i\underline{Z}_i - \underline{C}_i. \qquad (4.15)$$

Now, taking i=1, j=2, then i=2, j=1 in (4.15), a little algebraic manipulating yields

$$\begin{aligned} &\text{for } i=1,\ j=2; \quad Q_{11}\underline{M}_1 + Q_{12}\underline{M}_2 = \underline{R}_1, \\ &\text{for } i=2,\ j=1; \quad Q_{21}\underline{M}_1 + Q_{22}\underline{M}_2 = \underline{R}_2, \end{aligned} \qquad (4.16)$$

where

$$\begin{aligned} Q_{ii} &= K_i + (K_i')^{-1}E_{1i} + K_iL_{ij}(I-K_jL_{ji}K_iL_{ij})^{-1}K_jL_{ji}K_i, \\ Q_{ij} &= K_iL_{ij}(I-K_jL_{ji}K_iL_{ij})^{-1}K_j \\ \underline{R}_i &= \underline{U}_i - K_iL_{ij}(I-K_jL_{ji}K_iL_{ij})^{-1}(K_jL_{ji}K_i\underline{Z}_i+K_jL_{ji}\underline{C}_i \\ &\quad + K_j\underline{Z}_j+\underline{C}_j) - K_i\underline{Z}_i - \underline{C}_i \end{aligned} \qquad (4.17)$$

The value of $\underline{M}_1$ obtained by solving the linear system (4.16) is the optimal policy from G_{11}'s viewpoint when G_{12} uses $\underline{M}_2$, and vice versa, since (4.10) is satisfied.

Because of assumption 4.2.1 (perfect information), G_{11} and G_{12} can be envisaged as arriving at the system of equations (4.16) independently. Then, solving this system, G_{1i} determines its control policy M_i from the solution, disregarding M_j. There need be no communication between G_{11} and G_{12} during this procedure. We now investigate the effect of relaxing assumption 4.2.1.

4.3 An Iterative Procedure for Determining Optimal Equilibrium Strategies without Perfect Information

In this section the assumption of perfect information is replaced by

> Assumption 4.3.1. G_{1i} is aware of only that part of the causal subsystem which it affects; i.e. it knows (4.i) but not (4.j). Also, G_{1i} has no knowledge about G_{12}'s goal and vice versa.

Under assumption 4.3.1, G_{11} and G_{12} cannot determine equations (4.16), as they were portrayed as doing in the previous section. However, under certain conditions, the solution of (4.16) can be found by the iterative procedure* described below.

A temporal repetition of the finite process governed by (4.1) and (4.2) is imagined to occur as follows: G_{11} assumes $\underline{w}_i(t)=\underline{0}$ for t=1, 2, ..., T, where $\underline{w}_i(t)$ is given by (4.5). Using this particular sequence, which we denote by $(W_i)_1$, the operation $(M_i)_1$ corresponding to $(W_i)_1$ can be determined from (3.6),** reproduced here with appropriate notational changes.

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*Which can be thought of a "fictitious play"(16) of the game described in the preceding footnote.

**or from the dynamic programming solution of Appendix C.

$$(\underline{M}_i)_n = (K_i'K_i+E_{1i})^{-1}K_i' \; \underline{U}_i-K_i(\underline{W}_i)_n-C_i \, , \tag{4.18}$$

Then, the trajectory $(X_1)_1$ is obtained from (4.6), rewritten as

$$(\underline{X}_i)_n = K_i \; (\underline{W}_i)_n + (\underline{M}_i)_n \; + \underline{C}_i . \tag{4.19}$$

G_{12} is now given the information about $(X_1)_1$, from which it can determine $(W_2)_2$ using (4.7), that is,

$$(\underline{W}_i)_n = L_{ij}(\underline{X}_j)_{n-1} + \underline{Z}_i . \tag{4.20}$$

Next, G_{12} obtains $(M_2)_2$ and $(X_2)_2$ just as G_{11} arrived at its corresponding sequences above. Then G_{11} is given the information about $(X_2)_2$ from which it can determine $(W_1)_3$, and the whole procedure is repeated. Continuation* of this generates the sequences

$$(\underline{X}_1)_1, \; (\underline{X}_1)_3, \; (\underline{X}_1)_5, \; \cdots \tag{4.21}$$

$$(\underline{X}_2)_2, \; (\underline{X}_2)_4, \; (\underline{X}_2)_6, \; \cdots \, . \tag{4.22}$$

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*The iterative procedure that is carried out by G_{11} and G_{12} could be classified as "adaptive behavior" or as a "learning process." What each element G_{1i} does is to modify the vector parameters which are the elements of the sequence W_i at each iteration to take into account "new information" acquired during the previous iteration. The adaptive behavior ceases if the difference between adjacent $(W_1)_n$'s is less than some prescribed number δ .

We now prove

Theorem 4.3.1. A sufficient condition for the convergence of $(\underline{X}_i)_i$, $(\underline{X}_i)_{2+i}$, $(\underline{X}_i)_{4+i}$, ... is

$$||K_i(K_i'K_i+E_{1i})^{-1}E_{1i}L_{ij}K_j(K_j'K_j+E_{1j})^{-1}E_{1j}L_{ji}|| < 1. \quad (4.23)$$

Proof. Substitution of (4.18) into (4.19) yields

$$\begin{aligned}(\underline{X}_i)_n &= K_i(\underline{W}_i)_n+K_i(K_i'K_i+E_{1i})^{-1}K_i'\left[\underline{U}_i-K_i(\underline{W}_i)_n-\underline{C}_i\right] + \underline{C}_i \\ &= K_i\left[I-(K_i'K_i+E_{1i})^{-1}K_i'K_i\right](\underline{W}_i)_n + \underline{q}_i \ , \end{aligned} \quad (4.24)$$

where

$$\underline{q}_i = \underline{C}_i+K_i(K_i'K_i+E_{1i})^{-1}K_i'(\underline{U}_i-\underline{C}_i). \quad (4.25)$$

Substitution of (4.20) into (4.24), along with rewriting $I-(K_i'K_i+E_{1i})^{-1}K_i'K_i$ as $(K_i'K_i+E_{1i})^{-1}E_{1i}$ gives

$$(\underline{X}_i)_n = K_i(K_i'K_i+E_{1i})^{-1}E_{1i}L_{ij}(\underline{X}_j)_{n-1}+\underline{r}_i \ . \quad (4.26)$$

where

$$\underline{r}_i = \underline{q}_i+K_i(K_i'K_i+E_{1i})^{-1}E_{1i}\underline{Z}_i. \quad (4.27)$$

Now, if we substitute (4.26) with i replaced by j and n replaced by $n-1$ into (4.26) as it stands, we obtain the important result

$$\begin{aligned}(\underline{X}_i)_n = K_i(K_i'K_i+E_{1i})^{-1}E_{1i}L_{ij}K_j(K_j'K_j+E_{1j})^{-1} \\ E_{1j}L_{ji}(\underline{X}_i)_{n-2}+\underline{v}_i\end{aligned} \quad (4.28)$$

where

$$\underline{v}_i = \underline{r}_i+K_i(K_i'K_i+E_{1i})^{-1}E_{1i}L_{ij}\underline{r}_j \ . \quad (4.29)$$

It will be convenient in the sequel to have the abbreviations

$$II_i = K_i(K_i'K_i+E_{1i})^{-1}E_{1i}L_{ij}K_j(K_j'K_j+E_{1j})^{-1}E_{1j}L_{ji}, \tag{4.30}$$

$$\pi_i = (K_i'K_i+E_{1i})^{-1}E_{1i}. \tag{4.31}$$

Then, the mapping (4.28) from $(\underline{X}_i)_{n-2}$ to $(\underline{X}_i)_n$ becomes

$$(\underline{X}_i)_n = II_i(\underline{X}_i)_{n-2}+\underline{v}_i. \tag{4.32}$$

Subtracting $(\underline{X}_i)_{n-2}$ from $(\underline{X}_i)_n$ and taking the norm* of the result yields the distance from $(\underline{X}_i)_{n-2}$ to $(\underline{X}_i)_n$;

$$\begin{aligned} ||(\underline{X}_i)_n-(\underline{X}_i)_{n-2}|| &= ||II_i[(\underline{X}_i)_{n-2}-(\underline{X}_i)_{n-4}]|| \\ &\leq ||II_i||\cdot||(\underline{X}_i)_{n-2}-(\underline{X}_i)_{n-4}||. \end{aligned} \tag{4.33}$$

In other words,

$$d[(\underline{X}_i)_n,(\underline{X}_i)_{n-2}| \leq a_i d[(\underline{X}_i)_{n-2},(\underline{X}_i)_{n-4}], \tag{4.34}$$

where $a_i = ||II_i||$. Referring back to the discussion on contraction mappings in Chapter II, we see from (4.34) that $a_i < 1$ is a sufficient condition for the iterative procedure described above and the associated sequences (4.21) and (4.22) to converge. This concludes the proof of Theorem 4.3.1.

If the convergence criteria $||II_i|| < 1$ appearing in the statement of Theorem 4.3.1 are satisfied, the limits of the sequences (4.21) and (4.22) can be found by setting $(\underline{X}_i)_n = (\underline{X}_i)_{n-2}$ in (4.32); thus

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*as defined in Chapter III, equation (3.9).

$$(\underline{X}_i)_e = (I-II_i)^{-1}\underline{v}_i \qquad (4.35)$$

where $\underline{v}_i$ can be found from (4.25), (4.27), and (4.29). The subscript "e" above is used as a reminder that $(\underline{X}_i)_e$ is an equilibrial solution or "fixed point" of the mapping (4.32). Equilibrium values for $(\underline{W}_i)_e$ are found from (4.7) with $\underline{X}_j = (\underline{X}_j)_e$, and the equilibrium policy $(M_1)_e$ is given by (4.18) with "n" replaced by "e". We now prove

Theorem 4.3.2. The equilibrium vectors $(\underline{M}_1)_e$ and $(\underline{M}_2)_e$ determined in the iterative procedure described above satisfy equations (4.16).

Proof. The equilibrium vectors $(\underline{M}_i)_e$, $(\underline{W}_i)_e$, and $(\underline{X}_i)_e$ are governed by the same functional relationships as their counterparts $\underline{M}_i$, $\underline{W}_i$, and $\underline{X}_i$ in section 4.2. Hence, an identical argument,* except for the "$()_e$" around each sequence, can be used to arrive at a system of equations identical to (4.16) with $(\underline{M}_1)_e$ and $(\underline{M}_2)_e$ as the unknown.**

In summary, we have developed an iterative procedure to obtain the equilibrium policies without making

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*This argument must be carried out from the researcher's viewpoint, since under assumption 4.3.1 G_{1i} does not have the information necessary to derive (4.16).

**This can also be shown by direct substitution of $(\underline{M}_i)_e$ into (4.16), but this is extremely cumbersome algebraically.

the "perfect information" assumption. The goal-seeking elements G_{11} and G_{12} alternated in determining and applying their control actions. In true simultaneous control, G_{11} and G_{12} would be required to apply the control actions at the same time. The next section is devoted to the study of true simultaneous control.

4.4 Simultaneous First-Level Control with Incomplete Information

The purpose of this section is to investigate conditions under which an iterative procedure similar to the one derived in the previous section is useful in determining equilibrium policies under the "simultaneous control" requirement. Throughout this section, it will be assumed that G_{1i} possesses information only about the dynamics of the state variables under its cognizance,* i.e. G_{1i}'s control problem is "choose M_i so as to minimize (4.4) subject to

$$\underline{x}_i(t+1) = A_{ii}\underline{x}_i(t) + \underline{w}_i(t+1) + \underline{m}_i(t+1).\text{"} \qquad (4.36)$$

Although G_{1i} can measure the disturbance $\underline{w}_i(t)$, it does not know the mechanism producing $\underline{w}_i(t)$, i.e. equation (4.5).

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*An organizational interpretation is[22] "When tasks have been allocated to an organizational unit in terms of a subgoal, other goals and other aspects of the goals of the larger organization tend to be ignored in the decisions of the subunit."

The process to be controlled assumed to be an infinite-stage process which is periodic, with period T+1. The dynamics are governed by (4.1) and (4.2) with

$$\underline{z}_i(t) = \underline{z}_i(t+nT+n), \quad n = 0, 1, 2, \ldots . \tag{4.37}$$

Also, in (4.4),

$$\underline{u}_i(t) = \underline{u}_i(t+nT+n). \tag{4.38}$$

The initial conditions are imposed at T+1, 2T+2, ..., so that,

$$\underline{x}(nT+n) = \underline{c}, \; n = 0, 1, 2, \ldots . \tag{4.39}$$

Under this assumption, the finite T-stage process and the associated control problems formulated earlier are simply repeated an indefinite number of times.

Control of this process is assumed to proceed as follows: G_{1i} assumes $\underline{w}_i(t) = \underline{0}$ for t=1, 2, ..., T and obtains $(M_i)_1$* from (4.9) with $\underline{W}_i = \underline{0}$. It simultaneously implements $(M_i)_1$ and observes

$$\underline{w}_i(t+1) = \underline{x}_i(t+1) - A_{ii}\underline{x}_i(t) - m_i(t+1) \tag{4.40}$$

as the process evolves over t=0, 1, ..., T-1, according to (4.19) with n=1. At t=T, G_{1i} will compiled the sequence $(W_i)_1$ of "retrospective residuals". For n=1, 2, 3, ..., we use

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*As before, subscripts outside the parenthesis indicate the iteration or, in this case, the interval, over which the sequence is used.

Assumption 4.4.1. The anticipated* sequence of disturbances over the time interval nT+n, nT+n+1, ..., (n+1)T+n is $(M_i)_n$, the sequence of disturbances observed during the previous T-stage interval.

The control action to be applied during the $n^{\underline{th}}$ period is therefore computed from the formula

$$(\underline{M}_i)_n = (K_i'K_i+E_{1i})^{-1}K_i'\left[\underline{U}_i-K_i(\underline{W}_i)_{n-1}-\underline{C}\right], \qquad (4.41)$$

$$n=2,\ 3,\ 4,\ \ldots,$$

and the state trajectory $(X_i)_n$ is governed by (4.19).

The strict requirement of simultaneous control specifies that

$$(\underline{W}_i)_n = L_{ij}(\underline{X}_j)_n+\underline{Z}_i \qquad (4.42)$$

is the appropriate form of (4.7) to be used in this section, as opposed to the "lagged" version (4.20) of the last section.

If the sequence $(\underline{W}_i)_1$, $(\underline{W}_i)_2$, ..., $(\underline{W}_i)_n$, ... converges, the anticipatory mechanism of assumption 4.4.1 is accurate to any prescribed degree for n larger than some N which depends on the accuracy required. We now investigate the convergence of this sequence.

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*The reader may prefer to think of $\underline{W}_i$ as a stochastic vector. The prediction is then indicated by

$$(\underline{W}_i)_n = E\left[(\underline{W}_i)_{n+1} \mid (\underline{W}_i)_n,\ (\underline{W}_i)_{n-1},\ \ldots,\ (\underline{W}_i)_1\right],$$

$$n=1,\ 2,\ 3,\ \ldots$$

The discrepancy between the actual and anticipated disturbances over the $n^{\underline{th}}$ period is

$$(\Delta \underline{W}_i)_n = (\underline{W}_i)_n - (\underline{W}_i)_{n-1} \cdot \qquad (4.43)$$

Substitute this into (4.19) to get

$$(\underline{X}_i)_n = K_i\left[(\underline{W}_i)_{n-1} + (\Delta \underline{W}_i)_n + (\underline{M}_i)_n\right] + \underline{C}_i \qquad (4.44)$$

and substitute for $(\underline{M}_i)_n$ in (4.44) its equivalent vector as given by (4.41); then,

$$(\underline{X}_i)_n = K_i(\Delta \underline{W}_i)_n + K_i\left[I-(K_i'K_i+E_{1i})^{-1}K_i'K_i\right](\underline{W}_i)_{n-1} + \underline{q}_i \ , \qquad (4.45)$$

where $\underline{q}_i$ is given by (4.25). The argument continues along the lines of the argument from (4.24) to (4.29) in the previous section, so these steps are omitted. The important result is the functional form of the mapping from $(\underline{X}_i)_{n-2}$ to $(\underline{X}_i)_n$;

$$(\underline{X}_i)_n = K_i(K_i'K_i+E_{1i})^{-1}E_{1i}L_{ij}K_j(K_j'K_j+E_{1j})^{-1}E_{1j}L_{ji}(\underline{X}_i)_{n-2}$$
$$+K_i(\Delta \underline{W}_i)_n + K_i(K_i'K_i+E_{1i})^{-1}E_{1i}L_{ij}K_j(\Delta \underline{W}_j)_{n-1} + \underline{v}_i \ . \qquad (4.46)$$

From (4.46) and the notions of norms and distances discussed earlier,

$$d\left[(\underline{X}_i)_n, (\underline{X}_i)_{n-2}\right] = ||(\underline{X}_i)_n - (\underline{X}_i)_{n-2}||$$
$$\leqq ||\Pi_i||\cdot||(\underline{X}_i)_{n-2} - (\underline{X}_i)_{n-4}|| + ||K_i\left[(\Delta \underline{W}_i)_n - (\Delta \underline{W}_i)_{n-2}\right]$$
$$+ K_i\pi_i L_{ij}K_j\left[(\Delta \underline{W}_j)_{n-1} - (\Delta W_j)_{n-3}\right]||$$
$$= ||\Pi_i||\, d\left[(\underline{X}_i)_{n-2}, (\underline{X}_i)_{n-4}\right] + \delta_n \ , \qquad (4.47)$$

where II_i and π_i are given by (4.30) and (4.31), respectively. The positive term δ_n is small, being a function of second differences. Its lower bound is zero, hence in order for the inequality (4.47) to hold for all possible values of δ_n, we must have

$$d\left[(\underline{X}_i)_n,(X_i)_{n-2}\right] \leq ||II_i|| d\left[(X_i)_{n-2},(X_i)_{n-4}\right]. \qquad (4.48)$$

The control procedure described above establishes the successive application of (4.46) which, if $||II_i|| < 1$, because of (4.48), generates the convergent sequences

$$(\underline{X}_i)_1,\ (\underline{X}_i)_3,\ \cdots,\ (\underline{X}_i)_{2n-1},\ \cdots \qquad (4.49)$$

$$(\underline{X}_i)_2,\ (\underline{X}_i)_4,\ \cdots,\ (\underline{X}_i)_{2n},\ \cdots . \qquad (4.50)$$

We now prove

<u>Theorem 4.4.1.</u> If $||II_i|| < 1$, the sequences (4.49) and (4.50) both converge to $(\underline{X}_i)_e$, the limit of the sequence (4.2i) in the previous section.

Proof. Let $(\underline{X}_i)_e^{\,1}$ and $(\underline{X}_i)_e^{\,2}$ denote the limits of (4.49) and (4.50), respectively. Then, taking note of (4.42) and (4.43), we see from (4.46) that the following equations are satisfied, since $(\underline{X}_i)_e^{\,1}$ and $(\underline{X}_i)_e^{\,2}$ are also "fixed points" of the mapping (4.46) for n odd and even, respectively;

$$(\underline{X}_i)_e^{\,1} = (I-II_i)^{-1}\Big\{K_iL_{ij}\left[(\underline{X}_j)_e^{\,1}-(\underline{X}_j)_e^{\,2}\right.+\underline{v}_i+K_i\pi_iL_{ij}K_jL_{ji}$$
$$\left[(\underline{X}_i)_e^{\,2}-(\underline{X}_i)_e^{\,1}\right]\Big\},$$

$$(\underline{X}_i)_e^{\,2} = (I-\Pi_i)^{-1}\Big\{ K_iL_{ij}\left[(\underline{X}_j)_e^{\,2}-(\underline{X}_j)_e^{\,1}\right]+\underline{v}_i+K_i\pi_iL_{ij}K_jL_{ji}$$
$$\left[(\underline{X}_i)_e^{\,1}-(\underline{X}_i)_e^{\,2}\right]\Big\},$$

from which we obtain by subtraction

$$\left[I+2(I-\Pi_i)^{-1}K_i\pi_iL_{ij}K_jL_{ji}\right]\left[(\underline{X}_i)_e^{\,1}-(\underline{X}_i)_e^{\,2}\right]$$
$$= 2(I-\Pi_i)^{-1}K_iL_{ij}\left[(\underline{X}_j)_e^{\,1}-(\underline{X}_j)_e^{\,2}\right]. \qquad (4.48)$$

The two vector equations generated by (4.48) for i=1, j=2 and i=2, j=1 can only hold if

$$(\underline{X}_1)_e^{\,1}-(\underline{X}_1)_e^{\,2} = \underline{0}, \quad (\underline{X}_2)_e^{\,1}-(\underline{X}_2)_e^{\,2} = \underline{0},$$

so that (4.46) must reduce to (4.35) in a state of equilibrium. This concludes the proof of Theorem 4.4.1.

We have shown that the equilibrium state for the simultaneous control case of this section is identical to the equilibrium state of the simpler iterative procedure of the previous section. It follows that the equilibrium policies $(M_1)_e$ and $(M_2)_e$ are the same, and that $(\underline{M}_1)_e$ and $(\underline{M}_2)_e$ satisfy equations (4.16).

4.5 Sub-optimality of 1L2G Control

In this chapter we have been concerned with the deriviation of equilibrium policies, denoted by $(M_1)_e$ and $(M_2)_e$. Let $(M)_e$ and $(X)_e$ denote the over-all policy obtained by combining $(M_1)_e$ and $(M_2)_e$ and the resulting line of behavior,

respectively, so that, from (4.3),

$$g\left[(M)_e,(X)_e\right] = g_{11}\left[(M_1)_e,(X_1)_e\right] + g_{12}\left[(M_2)_e,(X_2)_e\right].$$

Notice, however, that the solution of (4.16) is not necessarily the solution of (3.6). Denote the latter by $(M)_o$; then

$$g\left[(M)_o,(X)_o\right] \leq g\left[(M)_e,(X)_e\right] \qquad (4.49)$$

since, by definition $(M)_o$ is the point at which $g(M,X)$ achieves its minimum subject to (3.1). In the next chapter, we introduce a "second-level" unit to "coordinate" the "action-counteraction" procedures of G_{11} and G_{12}, so that the equilibrium policy $(M)_e$ is also optimal.

CHAPTER V

SECOND LEVEL CONTROL IN A 2L3G SYSTEM

5.1 Introduction

The purpose of this chapter is to investigate the role of the second level goal-seeking element in a 2L3G controller (see Fig. 2.1). We will continue to use the control problem of Chapters III and IV. Having the results of these chapters available allows us to concentrate on the concepts relevant to second-level control. Slight changes in notation will be pointed out; that which is carried over will not be elaborated upon. When i and j are used as subscripts, $i=1, 2$, $j=1, 2$, and $i \neq j$, as before.

We list the assumptions which will hold in the sequel;

> Assumption 5.1.1. The partitioning of $\underline{x}(t)$, $\underline{m}(t)$, and $\underline{z}(t)$ which allows (3.1) to be rewritten as (4.1) and (4.2) is in effect.
>
> Assumption 5.1.2. The goal of the second level element G_2 is considered to be the over-all controller goal. This is to minimize (3.2) subject to (3.1).
>
> Assumption 5.1.3. The second level element G_2 alone has knowledge about the entire system. It knows the method of partitioning mentioned above, the dynamics of the entire causal subsystem, and the goals of the first level goal-seeking elements.

Assumption 5.1.4. The first-level element G_{1i} has the same information and control problem as in Section 4.4. The process being controlled is the infinite-stage periodic* one described in that section, and the control procedure employed by G_{11} and G_{12} is identical to the one discussed there. G_{1i}'s loss function is rewritten as**

$$g_{1i}(M_i,X_i) = \sum_{t=1}^{T} \underline{x}_i(t)-\underline{u}_i^*(t) \,'\, \underline{x}_i(t)-\underline{u}_i^*(t)$$

$$+ \sum_{t=1}^{T} \underline{m}_i'(t)D_{1i}^* \underline{m}_i(t), \qquad (5.1)$$

where D_{1i}^* is positive-definite and diagonal.

5.2 Synthesis of the Optimal Control Rule

In this section, G_2's control problem is "choose U_1* and U_2* in such a manner that G_{11} and G_{12} will together determine M so as to minimize g(M,X) subject to (3.1)." We will make the

---

*Recall the remarks made in Section 4.4; the finite-stage control problems just repeat in time. An "optimal" policy will be understood to be over a single T-stage period unless otherwise noted.

**The "separability" of g(M,X) as exemplified by (4.3) in the previous chapter does not, in general, hold here. In particular, usually we will have

$$D \neq \begin{bmatrix} D_{11}^* & 0 \\ 0 & D_{12}^* \end{bmatrix}.$$

Assumption 5.2.1. The convergence condition $||\Pi_i|| < 1$ in (4.48) is satisfied; thus, the control procedure engaged in by G_{11} and G_{12} tends toward a state of equilibrium.

We can apply the results of Chapter III to get the sequence of controlled input vectors which is optimal from G_2's viewpoint. This is (3.6), reproduced here for convenience,

$$(\underline{M})_o = (K'K+E)^{-1}K'(\underline{U}-\underline{C}-K\underline{Z}). \tag{5.2}$$

The equilibrium policies $(M_1)_e$ and $(M_2)_e$ are found in Chapter IV by replacing "n" by "e" in (4.18);

$$(\underline{M}_i)_e = (K_i'K_i+E_{1i}^*)^{-1}K_i'\left[\underline{U}_i^*-\underline{C}_i-K_i(\underline{M}_i)_e\right], \tag{5.3}$$

where

$$E_{1i}^* = \begin{bmatrix} D_{1i}^* & 0 & \cdots & 0 \\ 0 & D_{1i}^* & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & D_{1i}^* \end{bmatrix} \tag{5.4}$$

in accordance with earlier usage. Let

$$(\underline{M}_i)_o = \left[(K'K+E)^{-1}K'(\underline{U}-\underline{C}-K\underline{Z})\right]_i \tag{5.5}$$

denote the vector formed from the elements of $(\underline{M})_o$ corresponding to the controlled inputs under G_{1i}'s cognizance. In order for the equilibrium policies to agree with the optimal over-all policy, we must have

$$(\underline{M}_i)_o = (\underline{M}_i)_e \,.$$

Setting the right hand sides of (5.3) and (5.5) equal to

each other, one gets, after a little algebra,

$$\underline{U}_i^* = \underline{C}_i + K_i(\underline{W}_i)_e + (K_i')^{-1}(K_i'K_i + E_{1i}^*)\Big[(K'K+E)^{-1}K'$$

$$(\underline{U} - \underline{C} - K\underline{Z})\Big]_i \; . \qquad (5.7)$$

Equation (5.6) specifies a rule* for G_2 to follow in its act of determining the "ideal trajectories" U_1^* and U_2^* for the lower level units. With these "distorted" ideal trajectories, the equilibrium policies arrived at by G_{11} and G_{12} using the control procedure of Section 4.4 together determine the optimal policy.

Notice that G_2's rule for determining U_i^*, namely (5.7), depends on the equilibrium value $(\underline{W}_i)_e$. Because of assumption 5.1.3, this is available to G_2; the procedure for finding it goes as follows:

1. The trajectory $(X)_o$ under the optimal control law is determined from (3.3) with $\underline{M} = (\underline{M})_o$.

---

*It is interesting to note the geometrical effect of G_2's specification of $\underline{U}_i^*$ according to (5.7). Looking back to (4.16) and (4.17), it is easily seen that G_2 is adjusting the positions of the hyperplanes corresponding to the individual equations of (4.16), so that they interact at the point represented by $\left[(\underline{M}_1)_o, (\underline{M}_2)_o\right]$.

2. $(X)_o$ is partitioned into $(X_1)_o$ and $(X_2)_o$. Since these are also the trajectories determined by (4.1) and (4.2) under the equilibrium policies, $(W_i)_e$ is determined from (4.7):

$$(\underline{W}_i)_e = L_{ij}(\underline{X}_j)_o + \underline{Z}_i \ . \qquad (5.8)$$

The entire control procedure can now be envisaged to occur as follows.* Before getting underway, i.e. prior to t=0, G_2 determines the ideal trajectories for G_{11} and G_{12} from (5.7). Then, as the process evolves as described in section 4.4, the sequence of policies

$$(\underline{M}_i)_1, (\underline{M}_i)_2, \ldots, (\underline{M}_i)_n, \ldots, \qquad (5.9)$$

where $(\underline{M}_i)_n$ is applied from $(n-1)T+(n-1)$ to $nT+n$, approaches the limit $(M_i)_o$, as given by (5.5). The optimal policy for the infinite-stage process from G_2's viewpoint is, of course

$$(\underline{M}_i)_o, (\underline{M}_i)_o, \ldots, (\underline{M}_i)_o, \ldots, \ . \qquad (5.10)$$

Sequence (5.9) therefore represents an approximation to the optimal (infinite-stage) policy.

Again we remind the reader that our 2L3G system is contrived to preserve the autonomy of G_{11} and G_{12}; thus, we do not allow in the systems under consideration here, G_2 to "tell" G_{11} and G_{12} the $(W_i)_e$ determined in (5.8). Rather G_2 is forced to allow them to arrive at $(W_i)_e$ with no influence save U_i according to (5.7).

Let us examine the implication of the results of this section in regard to section 4.5 of the previous chapter. From that section, we can conclude, in the spirit of this chapter, that if G_2 transmits $U_i^* = W_i$ instead of "distorting" U_i according to (5.7), then the equilibrium control actions of the first level units will be suboptimal from G_2's viewpoint. Organizationally, this distortion represents a coordinative action.

5.3 The Effects of Varying E_{1i}^*

In this section we examine some aspects of allowing G_2 to adjust the elements of the matrix E_{1i}^* in $g_{1i}(M_i, X_i)$, as well as the ideal trajectories U_i^*.

The abbreviations,

$$\Pi_i = K_i(K_i'K_i+E_{1i}^*)^{-1}E_{1i}^*L_{ij}K_j(K_j'K_j+E_{1j}^*)^{-1}E_{ij}^*L_{ji} \tag{5.11}$$

$$\pi_i = (K_i'K_i+E_{1i}^*)^{-1}E_{1i}^*, \tag{5.12}$$

which are analogous to (4.30) and (4.31), are also made here. We begin by proving two theorems:

Theorem 5.3.1. The characteristic roots of the positive definite matrix $(K_i'K_i+E_{1i}^*)^{-1}E_{1i}^*$ all lie in the interval (0,1).

Proof. $K_i'K_i+E_{1i}^*$ is positive-definite, as we noted in Chapter III; hence, its inverse and, as a result, the product $(K_i'K_i+E_{1i}^*)^{-1}E_{1i}^* = \pi_i$ is positive-definite. Therefore,

the characteristic roots of π_i are all positive. We list two theorems from Bellman[7] as lemmas:

> Lemma 5.3.1. Let A and B be symmetric matrices, with B positive definite. Then
>
> $\mu_k(A) < \mu_k(A+B)$, $k=1, 2, \ldots, N$.

We remind the reader that $\mu_k(A)$ denotes the $k^{\underline{th}}$ characteristic root of A, and recall the convention stated earlier;

$\mu_N \leq \mu_{N-1} \leq \cdots \mu_2 \leq \mu_1$

> Lemma 5.3.2. If a symmetric matrix A is non-singular, the characteristic roots of A^{-1} are the reciprocals of the characteristic roots of A. In particular, if A is positive definite, $\mu_1(A) = 1/\mu_N(A^{-1})$.

Armed with these two lemmas, and noting that

$$\pi_i^{-1} = I+(E_{1i}^*)^{-1}K_i'K_i \;, \tag{5.13}$$

we see that the matrix on the left-hand side of (5.13) has characteristic roots all greater than unity, hence the characteristic roots of π_i are all less than unity. This concludes the proof of Theorem 5.3.1.

> Theorem 5.3.2. Simultaneously increasing (decreasing) the elements of the positive definite diagonal matrix E_{1i}^* increases (decreases) $||\pi_i||$. A decrease is constrained so as to maintain the positive definiteness of E_{1i}^*.

Proof. Denote the change in E_{1i}^* by E_{1i}^*, which is also positive definite. Then, the change from E_{1i}^* to $E_{1i}^* \pm$ E_{1i}^* induces the change

$$(K_i'K_i)^{-1}E_{1i}^{*} \qquad (K_i'K_i)^{-1}E_{1i}^{*} \pm (K_i'K_i)^{-1} \; E_{1i}^{*} \; . \qquad (5.14)$$

Consider the "+" sign in (5.14). From Lemma 5.3.1, clearly the characteristic root relationship

$$\mu_1 \; (K_i'K_i)^{-1}E_{1i}^{*} < \mu_1 \; (K_i'K_i)^{-1}E_{1i}^{*} + (K_i'K_i)^{-1} \quad E_{1i}^{*}$$

holds, which, from Lemma 5.3.2, yields

$$\mu_N \; (E_{1i}^{*} + \; E_{1i}^{*})^{-1}K_i'K_i \; < \mu_N \; (E_{1i}^{*})^{-1}K_i'K_i \; ,$$

so that

$$\mu_N \; I + (E_{1i}^{*} + \; E_{1i}^{*})^{-1}K_i'K_i \; < \mu_N \; I + (E_{1i}^{*})^{-1}K_i'K_i \; .$$

Looking back at (5.13), and again utilizing Lemma 5.3.2, we have

$$\begin{aligned} \mu_i(\pi_i) &= \mu_1 \; (K_i'K_i + E_{1i}^{*})^{-1}E_{1i}^{*} \\ &= \mu_1 \quad (E_{1i}^{*})^{-1}K_i'K_i + I \;^{-1} \quad < \mu_1 \quad (E_{1i}^{*} + \; E_{1i}^{*})^{-1} \\ &\quad K_i'K_i + I \;^{-1} \\ &= \mu_1 \; (K_i'K_i + E_{1i}^{*} + \; E_{1i}^{*})^{-1}(E_{1i}^{*} + \; E_{1i}^{*}) \; . \end{aligned}$$

The latter inequality, taken with the property that $||A|| = \mu_1(A)$ if A is positive-definite (see Chapter III), yields

$$\begin{aligned} &||(K_i'K_i + E_{1i}^{*})^{-1}E_{1i}^{*} \; || < ||(K_i'K_i + E_{1i}^{*} + \; E_{1i}^{*})^{-1} \\ &\qquad (E_{1i}^{*} + \; E_{1i}^{*}) \; || \; ; \end{aligned}$$

thus, we have demonstrated the theorem for the word "increase." Now consider the "-" sign in (5.14). Note that, from Lemma 5.3.1 and the fact that E_{1i}^{*} remains positive definite after

the change, that

$$\mu_k(E_{1i}*-\Delta E_{1i}*) < \mu_k(E_{1i}*-\Delta E_{1i}*+\Delta E_{1i}*) = \mu_k(E_{1i}*) .$$

Using this result, the proof for the word "decrease" is exactly the same as for the word "increase", except for the reversal of the inequality at each step and a "-" before $\Delta E_{1i}*$. The result is

$$||(K_i'K_i+E_{1i}*-\Delta E_{1i}*)^{-1}(E_{1i}*-\Delta E_{1i}*)|| < ||(K_i'K_i+E_{1i}*)^{-1}E_{1i}*|| = ||\pi_i|| ,$$

completing the proof of Theorem 5.3.2. We now proceed to show how G_2 can use its capability of manipulating the elements of $E_{11}*$ and $E_{12}*$ to assure convergence of the first-level control process.

From (3.12), there exists a mathematical principle by which we can bound the contraction factor $||II_i||$ in (4.48) from above:

$$||II_i|| \leq ||K_i||\cdot||\pi_i||\cdot||L_{ij}||\cdot||K_j||\cdot||\pi_j||\cdot||L_{ji}|| . \tag{5.15}$$

By Theorem 5.3.1 the factors $||\pi_i||$ and $||\pi_j||$ in the right hand side of (5.15) are both less than unity. Moreover, by Theorem 5.3.2, G_2 can adjust $E_{11}*$ and $E_{12}*$ so as to make these factors small enough to assure $||II_i||<1$, i.e. to assure that the sequence (5.9) converges. This is important, for without convergence at the first level, there is no hope

of G_2 effecting any sort of near optimal control with (5.7).

5.4 Summary

In the two previous sections we have examined the relative effects of G_2 varying the parameters U_i^* and D_{1i}^* in G_{1i}'s loss function (5.1). In general, we have seen that D_{1i}^* affects the rate of convergence of the sequences (5.9), (4.49), (4.50), and others, while $\underline{U}_i^*$ determines the limits of these sequences. Both of these variations determine how well the sequence (5.9) approximates the control law. Notice that, whatever the choice* of E_{1i}^*, determination of U_i^* according to (5.7) assures that the limit of the sequence (5.9) is $(M_i)_o$.

We point out the following rather obvious point: Suppose G_2's control problem is to determine U_i^* and select D_{1i}^* from the set d^iI, where d^i lies in the interval (d_o^i, d_I^i), $d_o^i > 0$, in such a way to cause the sequence (5.9) to approximate the optimal control sequence (5.10) as well as possible. Then, d_o^iI is the proper choice for D_{1i}^* and $\underline{U}_i^*$ is then determined from (5.7) with

---

*or, if E_{1i}^* is held constant

$$E_{1i}^{*} = \begin{bmatrix} d_o{}^i I & 0 & \cdots & 0 \\ 0 & d_o{}^i I & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & d_o{}^i I \end{bmatrix} .$$

This assures the most rapid convergence of (5.9) to its limit (because of the method of choosing U_i^{*}), the vector $(\underline{M}_i)_o$. Appendix D consists of a description of a numerical example of how a variation in D_{1i}^{*} of this type effects $||II_i||$.

CHAPTER VI

SELF-ORGANIZATIONAL ASPECTS OF 2L3G SYSTEMS

6.1 Ordering Relations on Structure Space

In this chapter, we develop an ordering relation over a subset of the structures of the 2L3G controller which was studied in the previous chapter. The ordering relation will be linked with the following control problem, stated earlier in Chapter II; "Given a dynamical system

$$(\underline{X})_{n+1} = \left[Q_a \; (\underline{X})_n\right] \tag{6.1}$$

where the ransformation Q_a depends on the structure and has a contraction factor a, find the structure such that, given an initial value $(\underline{X})_1$, the system reaches equilibrium in the shortest possible time." The contraction factor a of Q_a, when associated with the element of the structure set which induces Q_a, yields a numerical measure of that structure. According to the control problem stated above and the earlier discussion (Chapter II) on contraction mappings, the smaller this a, the greater the effectiveness of the controller structure. This relationship between the value of the contraction factor and the effectiveness of the structure allows us to rank each element of the structure set according to the associated numerical value of a.

The criterion for choosing an element of the structure set will be based on the rate of convergence of the first-level control procedures of sections 4.3 and 4.4. In section 5.3, we showed the dependence of this convergence rate on the matrix parameter ${D_{1i}}^*$ of G_{1i}'s loss function (5.1); thus, we impose

> <u>Assumption 6.1.1.</u> The matrix parameters ${D_{11}}^*$ and ${D_{12}}^*$ in the first level units' loss functions will be held constant.*

This assumption will allow us to study the effects of structural changes alone on the rate of convergence.

As in Chapters IV and V, i=1, 2, j=1, 2, and $i \neq j$. Also in addition to the above assumption, Assumptions 5.1.1 through 5.1.4 hold throughout this chapter.

6.2 Controller Structure and Structural Change

Let us recall the statements of the control problems of the 2L3G system of the previous chapter:

1. G_{1i}: "Determine M_i so as to minimize (5.1) subject to (4.36)."

---

*Although, as we pointed out in a footnote in Chapter V, in general

$$D \neq \begin{bmatrix} {D_{11}}^* & 0 \\ 0 & {D_{12}}^* \end{bmatrix}$$

Organizationally, G_{1i} may attach different "weights" between the elements of $\underline{m}_i(t)$ than does G_2.

2. G_2: "Determine the sequences U_1^* and U_2^* which will cause G_{11} and G_{12} to determine M so as to minimize (3.2) subject to (3.1)."

When the control problems stated above are considered collectively, they characterize the organizational structure of the controller in the sense of section 1.1. We now state:

Definition 6.2.1. A member of the set of structures considered here is an identical allocation of the elements* of $\underline{x}(t)$, $\underline{m}(t)$ and $\underline{z}(t)$ to the control problems of G_{11} and G_{12} stated above. With each member of the structure set we can associate a "selector vector", $\underline{\sigma} = [\sigma_1, \sigma_2, \ldots, \sigma_s]'$, which specifies how this allocation is made as follows:

$\sigma_k = i$, $i=1, 2$, $k=1, 2, \ldots, s$ if $x_k(t)$, $m_k(t)$, and $z_k(t)$ are under the cognizance** of G_{1i}.

---

*Mesarović[17] defines "systems structure" as the set of relationships which, together with the system's terms (i.e. state variables, coefficients, etc.), define the system. If we introduce the set of relationships $[r_{hk}]$, where "$x_h(t)$ r_{hk} $x_k(t)$" means "$x_h(t)$ and $x_k(t)$ are under the cognizance of the same/different (whichever is appropriate under that particular allocation) first level goal-seeking element," then it is clear that every allocation of elements to G_{11} and G_{12} determines a unique set of relationships $[r_{hk}]$. The set $[r_{hk}]$ is, of course, only a subset of the total number of relationships between the terms. For example, in addition to $[r_{hk}]$, there are the relationships "=", "+", etc., included in (4.1) and (4.2).

**A slight conceptual difficulty arises from the fact that in (4.36), $\underline{z}_i(t+1)$ was lumped, along with $A_{ij}\underline{x}_j(t)$, into $\underline{w}_i(t+1)$. However, the control process of section 4.4 is identical whether G_{1i} is allowed specific knowledge of $\underline{z}_i(t)$ or not.

Henceforth, we will refer to an allocation of the elements of $\underline{x}(t)$ only; within the framework of the control problem considered here, this induces a similar allocation of $\underline{m}(t)$ and $\underline{z}(t)$ by way of the partitioning of (3.1) into (4.1) and (4.2).

> Definition 6.2.2. A change in organizational structure is simply a change in the allocation;* this induces a transformation on the selector vector.

For example, a "change of structure" would occur if, say, $x_k(t)$ were somehow moved from G_{11}'s cognizance to G_{12}'s; this would cause a change of the control functions of all the goal-seeking elements, although the 2L3G system configuration remains the same. G_2 would transmit $u_k^*(t)$ to G_{12} instead of G_{11}, and G_{11} would no longer be concerned with monitoring $x_k(t)$ or making the adjustment $m_k(t)$; these would be performed by G_{12}. This particular change would induce the transformation $1 \rightarrow 2$ on σ_k, the other elements of $\underline{\sigma}$ remaining unchanged.

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*Which induces a change in the set of relationships $[r_{hk}]$. Structural changes which would alter the mathematical relationships, such as replacing "=" by "<", are not considered. Neither are changes in configuration where, say, one of the first level goal-seeking elements are removed.

Consider a change of the type

$$(\sigma_k=1,\ \sigma_p=2)\longrightarrow(\sigma_k=2,\ \sigma_p=1); \tag{6.2}$$

i.e. "change $x_k(t)$ from $\underline{x}_1(t)$ to $\underline{x}_2(t)$, and change $x_p(t)$ from $\underline{x}_2(t)$ to $\underline{x}_1(t)$." This also causes similar changes between the control input vectors $\underline{m}_1(t)$ and $\underline{m}_2(t)$, the disturbance vectors $\underline{z}_1(t)$ and $\underline{z}_2(t)$, and the ideal trajectories $\underline{u}_1{}^*(t)$ and $\underline{u}_2{}^*(t)$. This change induces a row-column interchange in the matrix A as well:

$$\left[\begin{array}{ccc|ccc}
a_{11} & \cdots & a_{1k} & \cdots\ a_{1p} & \cdots & a_{1s} \\
\vdots & & \vdots & \vdots & & \vdots \\
a_{k1} & \cdots & a_{kk} & \cdots\ a_{kp} & \cdots & a_{ks} \\
\hline
a_{p1} & \cdots & a_{pk} & \cdots\ a_{pp} & \cdots & a_{ps} \\
\vdots & & \vdots & \vdots & & \vdots \\
a_{s1} & \cdots & a_{sk} & \cdots\ a_{sp} & \cdots & a_{ss}
\end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccc|ccc}
(a_{11})_{11} & \cdots & (a_{1p})_{1k} & \cdots\ (a_{1k})_{1p} & \cdots & (a_{1s})_{1s} \\
\vdots & & \vdots & \vdots & & \vdots \\
(a_{p1})_{k1} & \cdots & (a_{pp})_{kk} & \cdots\ (a_{pk})_{kp} & \cdots & (a_{ps})_{ks} \\
\hline
(a_{k1})_{p1} & \cdots & (a_{kp})_{pk} & \cdots\ (a_{kk})_{pp} & \cdots & (a_{ks})_{ps} \\
\vdots & & \vdots & \vdots & & \vdots \\
(a_{s1})_{s1} & \cdots & (a_{sp})_{sk} & \cdots\ (a_{sk})_{sp} & \cdots & (a_{s1})_{s1}
\end{array}\right]$$

The heavy dashed lines indicate the partitioning of A induced by the controller structure. The subscripts inside (outside) the parentheses indicate position in the matrix before (after) the change. There is, therefore, an interchange of elements

between the matrices A_{11}, A_{12}, A_{21}, and A_{22} as a result of the change in controller structure indicated by (6.2). For example, in A_{12}, the $p^{\underline{th}}$ column is transformed as

$$\begin{bmatrix} a_{1p} \\ a_{2p} \\ \vdots \\ a_{s_1,p} \end{bmatrix} \rightarrow \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{s_1,k} \end{bmatrix}$$

and the $k^{\underline{th}}$ row suffers the transition

$$(a_{k,s_1+1} \; a_{k,s_1+2} \cdots a_{k,s}) \longrightarrow (a_{p,s_1+1} \; a_{p,s_1+2} \cdots a_{ps}),$$

where s_i is the number of elements in $\underline{x}_i(t)$. We now proceed to indicate how a structural change such as the one described above affects the contraction factor $||\Pi_i||$ of the mapping (4.48) associated with the first-level control procedure of section 4.4.

6.3 Evaluation of the Effects of a Change in Structure

In this section, we shall discuss the dependence of the transformation Q_a in (6.1) on the structure under which the controller is operating.

Note that $||\Pi_i||$ in (4.48) is the norm of a product of several matrices (see 4.30), each of which is altered by a structural change. Looking back to the upper bound of $||\Pi_i||$ established by (5.15), we see that a reasonable

starting point of the analysis of this section is to examine how changes in $K_i'K_i$ affect $||\pi_i||$, where π_i is given by (5.12); thus, we prove the three theorems:

<u>Theorem 6.3.1</u>

$$\frac{||E_{1i}^*||}{||K_i'K_i||+||E_{1i}^*||} \leq ||(K_i'K_i+E_{1i}^*)^{-1}E_{1i}^*|| \; .$$

Proof: From (3.12),

$$||E_{1i}^*|| = ||(K_i'K_i+E_{1i}^*)\;(K_i'K_i+E_{1i}^*)^{-1}E_{1i}^*||$$

$$\leq (||K_i'K_i||+||E_{1i}^*||)\;||(K_i'K_i+E_{1i}^*)^{-1}E_{1i}^*||,$$

from which Theorem 6.3.1 follows.

<u>Theorem 6.3.2</u> If the elements of E_{1i}^* are identical, and equal to e_{1i}^*, then

$$||(K_i'K_i+E_{1i}^*)^{-1}E_{1i}^*|| \quad = \quad \frac{e_{1i}^*}{e_{1i}^*+\mu_N(K_i'K_i)}. \qquad (6.3)$$

Proof: Let P be an orthogonal matrix such that

$$K_i'K_i = P \begin{bmatrix} \mu_1 & & & 0 \\ & \mu_2 & & \\ & & \ddots & \\ 0 & & & \mu_N \end{bmatrix} P'$$

Then according to Bellman[7]

$$(K_i'K_i+E_{1i}^*)^{-1}E_{1i}^* = (K_i'K_i+e_{1i}^*I)^{-1}e_{1i}^*I$$

$$= P \begin{bmatrix} (\mu_1+e_{1i}^*)^{-1}e_{1i}^* & & & \\ & (\mu_2+e_{1i}^*)^{-1}e_{1i}^* & & \\ & & \ddots & \\ & & & (\mu_N+e_{1i}^*)^{-1}e_{1i}^* \end{bmatrix} P'.$$

The diagonal elements in the matrix immediately above are the characteristic roots of $(K_i'K_i+E_{1i}^*)^{-1}E_{1i}^*$, so the theorem follows immediately from property (3.17).

Theorem 6.3.3 An increase (decrease) in the smallest characteristic root of $(E_{1i}^*)^{-1}K_i'K_i$ results in a decrease (increase) in $||(K_i'K_i+E_{1i}^*)^{-1}E_{1i}^*||$

Proof: Let P be an orthogonal matrix such that

$$(E_{1i}^*)^{-1}K_i'K_i = P\begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & 0 \\ & 0 & \ddots & \\ & & & \mu_N \end{bmatrix} P'.$$

As in the proof of Theorem 6.3.2,

$$I+(E_{1i}^*)^{-1}K_i'K_i = P\begin{bmatrix} 1+\mu_1 & & & \\ & 1+\mu_2 & & 0 \\ & 0 & \ddots & \\ & & & 1+\mu_N \end{bmatrix} P'$$

so, an increase (decrease)in

Lemma 5.3.2 and equation (5.13), a decrease (increase) in

$$\mu_1|(K_i'K_i+E_{1i}^*)^{-1}E_{1i}^*| = ||(K_i'K_i+E_{1i}^*)^{-1}E_{1i}^*|| = ||\pi_i||,$$

Q.E.D.

In the previous section, the manner in which a structural change induced a reshuffling of the elements of A between A_{11}, A_{12}, A_{21}, and A_{22} was examined. Recalling

the definitions of K_i and L_{ij}, we see that a structural change would alter the elements of these matrices, with a subsequent change in the scalars $||K_i||$ and $||L_{ij}||$. Also, from the theorems proved immediately above,* the resulting change in $K_i'K_i$ affects the number $||\pi_i||$. Therefore, the factors in the upper bound (5.15) of $||\Pi_i||$ all change when the structure is altered. Theorem 6.3.1 illustrated how a decrease in $||K_i'K_i||$ forces a lower bound of $||\pi_i||$ to increase. Although the inequality expressed by Theorem 6.3.1 is somewhat weak, it illustrates that decreasing $||K_i'K_i||$ should be avoided if one were attempting to decrease $||\pi_i||$.

Theorem 6.3.2 gives us some insight as to how $||\pi_i||$ varies with changes in the minimum characteristic root of $K_i'K_i$. Because of the relationship expressed by Lemma 5.3.1, Theorem 6.3.2 can be of value in studying the effects of structural changes on $||\pi_i||$. Theorems 6.3.2, 5.3.1, and 5.3.2 enforce the intuitive notion that positive definite matrices are generalizations of positive numbers. Carrying this intuition a little farther, we can examine

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*and the fact that $\mu_N(K_i'K_i)$ is equal to $|K_i'K_i|$, the determinant of $K_i'K_i$, hence is a function of all the elements in $K_i'K_i$.

(4.30) and (5.15) and conclude that, if it were possible to effect a structural change which would cause only a "slight" change in the produce $||K_1||\cdot||K_2||\cdot||L_{12}||\cdot||L_{21}||$ while increasing $||K_i'K_i||$ considerably, this would be a "change for the better," as it would tend to decrease $||\pi_i||$ and hence an upper bound of $||\Pi_i||$.

Theorem 6.3.3 is an attempt to resolve the apparent narrow application of Theorem 6.3.2, due to the requirement that all the diagonal elements of $E_{1i}{}^*$ are identical. These two theorems therefore complement each other, Theorem 6.3.2 giving an insight as to the numerical behavior of $||\pi_i||$ as $\mu_N(K_i'K_i)$ varies, and Theorem 6.3.3 providing the generality to make this insight useful.

While it is possible to numerically determine the effects of the changes in the factors in the upper of $||\Pi_i||$ expressed by (5.15), it is more fruitful to calculate $||\Pi_i||$ directly when the computer is resorted to, since this only involves the computation of one maximum characteristic root instead of the six necessary to determine the right-hand side of (5.15).

In Chapter IV, we found the mapping between $(\underline{X}_i)_n$ and $(\underline{X}_i)_{n+2}$ and the conditions under which this mapping

is a contraction mapping, namely $||\Pi_i|| < 1$. The following theorem allows us to easily determine the contraction factor for the mapping between $(\underline{X})_n$ and $(\underline{X})_{n+2}$, the over-all system state trajectories over the $n^{\underline{th}}$ and $(n+2)^{\underline{th}}$ T-stage control periods, respectively, once $||\Pi_1||$ and $||\Pi_2||$ are known.

> Theorem 6.3.4 The contraction factor a of the mapping between $(\underline{X})_n$ and $(\underline{X})_{n+2}$ induced by the iterative procedures between G_{11} and G_{12} described in sections 4.3 and 4.4 is equal to the maximum of the contraction factors $||\Pi_1||$ and $||\Pi_2||$ of the mappings between $(\underline{X}_i)_n$ and $(\underline{X}_i)_{n+2}$ for i=1, 2, respectively.

Proof: Let

$$\Pi = \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix}.$$

From (3.15),

$$||\Pi_i|| = \left[\mu_1(\Pi_i'\Pi_i)\right]^{1/2}.$$

Let Q_i be an orthogonal matrix such that

$$\Pi_i'\Pi_i = Q_i \begin{bmatrix} \mu_1 & & 0 \\ & \mu_2 & \\ & & \ddots & \\ 0 & & & \mu_{s_iT} \end{bmatrix} Q_i' ;$$

then

$$\Pi'\Pi = \begin{bmatrix} \Pi_1'\Pi_1 & 0 \\ 0 & \Pi_2'\Pi_2 \end{bmatrix}$$

$$= \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} \mu_1(II_1'II_1) & & & & & 0 \\ & \ddots & & & & \\ & & \mu_{s_1T}(II_1'II_1) & & & \\ & & & \mu_1(II_2'II_2) & & \\ & & & & \ddots & \\ & 0 & & & & \mu_{s_2T}(II_2'II_2) \end{bmatrix} \begin{bmatrix} Q_1' & 0 \\ 0 & Q_2' \end{bmatrix} \quad (6.4)$$

where

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$$

is also an orthogonal matrix. Looking at (6.4) it is easy to see that

$$a = ||II|| = [\mu_1(II'II)]^{1/2} = \max\left\{[\mu_1(II_1'II_1)]^{1/2}, [\mu_1(II_2'II_2)]^{1/2}\right\}$$

from which the theorem follows.

Theorem 6.3.4 is convenient from a computational viewpoint, since it allows us to use the smaller matrices II_i in the maximum characteristic root algorithm[10] instead of the full-size matrix II. Also, the factors $||II_i||$ are of interest; after these are obtained, because of Theorem 6.3.4, no additional computing is required to obtain the "over-all" contraction factor.

The control procedure for the 2L3G system under study here is reviewed at this point: an iterative procedure is induced by the "interplay" between G_{11} and G_{12}. If $||II_i||<1$, this procedure converges in such a manner so that

the "equilibrium control actions" of G_{11} and G_{12} are optimal from G_2's viewpoint. This is accomplished through G_2's "influence" via adjustment of the parameters $\mu_i{}^*(t)$ in G_{1i}'s loss function according to (5.7). Ideally, G_2's optimal state trajectory would be

$$(\underline{X})_o, \ (\underline{X})_o, \ (\underline{X})_o, \ \cdots, \tag{6.5}$$

where $(X)_o$ is the T-stage sequence of state variables under the optimal control law $(M)_o$, obtained from (3.6). However, the state trajectory under the collective control action exerted by G_{11} and G_{12} turns out to be

$$(\underline{X})_1, \ (\underline{X})_2, \ (\underline{X})_3, \ \cdots, \tag{6.6}$$

a sequence whose limit, due to G_2's influence, is $(\underline{X})_o$. The faster (6.6) converges, the better (6.6) approximates (6.5). Theorem 6.3.4 and equation (4.30) allow us to compute the contraction factor of (6.6) for a specific element of the structure set, i.e. allocation of state variables between G_{11} and G_{12}.

Suppose two different structures were being compared. Further, assume that the contraction factor $||\Pi||$ is smaller for one structure than for the other. Then, clearly we could classify that structure with the smaller $||\Pi||$ as the "better" of the two, since (6.6) would converge faster when the 2L3G system was operating under that structural

configuration, resulting in a better approximation to (6.5).

The next section is concerned with a specific example illustrating the variation of $||II||$ as the structure is changed.

6.4 Self-Organizational Aspects

In this section, we make

> Assumption 6.4.1. The second level goal-seeking element G_2 in the 2L3G system under study here has the capability to change the structure of the controller within the framework specified by Definition 6.2.2, in addition to the capability of setting $\underline{u}_i^*(t)$ in G_{1i}'s loss function.

This assumption and the remarks made previously in this chapter compose the formulation of a "structural choice" control problem for G_2, similar to the one stated at the beginning of this chapter. For every element of the structure set, the 2L3G control procedure described in the previous section holds. No matter what structure the 2L3G system operates under, the sequence (6.6) converges to $(\underline{X})_o$. This is due to the capability of G_2 to manipulate $\underline{u}_i^*(t)$, which it does according to (5.7); thus, under any structural configuration the limit point of (6.6) is $(\underline{X})_o$. We now make

> Conjecture 6.4.1. For the 2L3G system studied here, there is a different value of $||II||$, the contraction factor of the mapping between $(\underline{X})_n$ and $(\underline{X})_{n+2}$ induced by the 2L3G control procedure, for each element of the structure set considered.

We state this as a conjecture rather than a theorem because of the immensely cumbersome algebra* associated with the proof of the statement. We cite the remarks of the previous section as evidence as to the plausibility of conjecture 6.4.1. Operationally, this conjecture can be proven true or false for any special case of the system studied here, by numerically evaluating $||\Pi||$ for every possible structure. For example, let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 3 & 4 & 2 & 1 & 1 \\ 2 & 1 & 4 & 3 & 3 \\ 1 & 1 & 1 & 6 & 2 \\ 2 & 8 & 2 & 3 & 8 \end{bmatrix} \qquad (6.7)$$

in (3.1) and $D_{11}^{*}=I$ and $D_{12}^{*}=I$ in (5.1). Consider the $\binom{5}{2}=10$ possible structures with two state variables in $\underline{x}_1(t)$ and three state variables in $\underline{x}_2(t)$. Table VI-1 shows the results of a computation using the definitions of K_i and L_{ij} implicit in (4.6) and (4.7), (4.30), (5.4) and Theorem 6.3.4. In this example, T=3. It is seen that only four of the ten structures would be considered "feasible" i.e. have $||\Pi||<1$, for the

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*For the norm considered here, one would have to determine $\mu_1(\Pi_i'\Pi_i)$ as a function of the elements of $\pi_1,\pi_2,K_1,K_2,L_{12}$, and L_{21} multiplied together in the appropriate order.

assumed values of D_{1i}^*. These are structures 3, 7, 9, and 10.

TABLE VI-1

structure	selector vector	$\|\|\Pi\|\|$
1	(1 1 2 2 2)	1.879
2	(1 2 1 2 2)	1.026
3	(1 2 2 1 2)	.935
4	(1 2 2 2 1)	1.432
5	(2 1 1 2 2)	1.251
6	(2 1 2 1 2)	1.376
7	(2 1 2 2 1)	.692
8	(2 2 1 1 2)	1.331
9	(2 2 1 2 1)	.846
10	(2 2 2 1 1)	.808

We will classify G_2's choice of a structure for the 2L3G system as "self-organizational activity." One type of self-organizational activity is for G_2 to determine a table similar to Table VI-1; then, if Conjecture 6.4.1 is found to hold, select that structure for which $||\Pi||$ is a minimum.

It is interesting to note, from (4.30), that $||\Pi||$ is independent of the uncontrollable variables U and Z. Consider the following situation: our 2L3G controller, concerned with guiding the causal subsystem so as to minimize (3.2) over each time period subject to (3.1) with A given by (6.7), has carried out the control procedure outlined in the previous section with the additional act by G_2 of determining the "best" structure from Table VI-1. As the process evolves,

(6.6) tends toward its limit $(\underline{X})_o$ at the fastest rate possible within the restrictions imposed by the structure set considered. The effect of an alternation* in U or Z is to change the optimal control law (see (3.6)), with a resulting shift in the "influences" U_1^* and U_2^* (according to (5.7)) and the equilibrium policies arrived at by G_{11} and G_{12} under these influences. However, no "reorganization", i.e. structural change by G_2, is necessary, because of the functional independence of $||\Pi||$ on U and Z. When a change in U or Z occurs, we can imagine t as being set equal to zero with the control procedure in question starting over from the beginning, with the exception that G_2 need not redetermine the optimal structure. The rate of convergence towards equilibrium, therefore, remains the same.

Suppose instead that the elements of the matrix A are subject to change, this change being immediately detectable by G_2. Then G_2 is forced to re-examine the contraction factor for all possible structures. A specific example serves to illustrate this point. Suppose, in (6.7), a_{52} and

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*Organizationally, a change in the ideal trajectory U might be termed a change in the objectives of the organization, while a change in Z would correspond to an environmental alteration.

a_{55} are abruptly changed from 8 to 4. Table VI-2 shows the results of a computation similar to the one that led to Table VI-1. According to these tables, G_2 should leave $x_2(t)$ under G_{12}'s control and shift each of the other state variables* from its existing structural location to the alternate

TABLE VI-2

structure	$\|\|I\|\|$
1	1.446
2	.835
3	.773
4	1.473
5	1.235
6	1.182
7	.912
8	1.225
9	.795
10	.817

one. After this structural change is effected, the control procedure evolves exactly as before. From the researcher's viewpoint, this activity appears as teleological self-organization; the structural configuration of the 2L3G controller is altered in response to a change in A in order to insure that (6.6) continues to approximate (6.5) as well as possible. The structure choice problem stated in section 6.1 is solved,

*The change from structure 7 to structure 9 might be more plausible if a "cost" is associated with each alteration. This would involve G_{11} and G_{12} "trading" $x_2(t)$ and $x_3(t)$ only.

since that structure which causes (6.6) to be the best approximation to (6.5) is also that structure that brings the system into equilibrium as quickly as possible.

One more example serves to illustrate these ideas further. Organizationally, this might be termed "reorganizing after a merger." Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 1 & 1 & 1 & 6 \end{bmatrix} \qquad (6.8)$$

in (3.1), and suppose G_{11} and G_{12} each control two state variables. Table VI-3 indicates that G_{11} should control $x_2(t)$ and $x_4(t)$. It is interesting to note that, for an

TABLE VI-3

structure	selector vector	$\|\|\Pi\|\|$
1	(1 1 2 2)	1.006
2	(2 1 2 1)	.690
3	(1 2 2 1)	.959

even number of state variables, in this case four, it is necessary to consider only $(1/2)\binom{4}{2}$ possible structures, since nothing is gained by calculating, say, $||\Pi||$ for (2 2 1 1). To continue, suppose $x_5(t)$ is added to the state vector, its interactions with the other four state variables and itself being determined by the fifth row and column of

A in (6.7). Table VI-1 indicates that G_{12} should take over $x_4(t)$ from G_{11} and the latter should incorporate $x_5(t)$ into its bailiwick.

6.5 Self-Organizational Activity from the State-Transition[6] Viewpoint

In Appendix C we derive the dynamic programming solution of the control problem of Chapter III. This solution is numerically equivalent to that given by (3.6), but the philosophy of implementation is somewhat different. While (3.6) yields the sequence of control actions "all at once" so that these actions are known before the process begins, the dynamic programming solution yields a formula of the type

$$\underline{m}(t+1) = \underline{m}\left[\underline{x}(t), \underline{u}(t), \underline{z}(t+1)\right] \; ; \tag{6.9}$$

thus, from the state-transition viewpoint, the control actions are not determined until the information on $\underline{x}(t)$ is known, i.e. immediately before the action is to be taken. The advantage of the dynamic programming solution was mentioned in section 3.5.

We now prove

Theorem 6.5.1 Let a mapping from $\underline{p}$ to $\underline{q}$ be defined by

$$\underline{q} = B\underline{p} + \underline{e} \; , \tag{6.10}$$

where B is a non-singular matrix. Then a necessary condition for (6.10) to be a contraction mapping is $||B|| < 1$.

Proof: If (6.10) is a contraction mapping, for two points $\underline{p}_1$ and $\underline{p}_2$,

$$||B|| \; ||\underline{p}_1-\underline{p}_2|| \leq ||B\underline{p}_1-B\underline{p}_2|| = ||q_1-q_2|| < ||p_1-p_2|| ,$$

hence

$$||B|| < 1$$

Q.E.D. Theorem 6.5.1 can be thought of as a companion theorem to Theorem 4.4.1 for linear mappings, since the two assert that $||B||<1$ is a necessary and sufficient condition for (6.10) to be a contraction mapping.

Suppose that T, the number of state transitions in each period, is large enough so that G_2 cannot apply (3.6) and (5.7), nor compute $||II||$ as indicated in the previous section, all because of the large size of the matrices in these formulas. Assume G_2 selects a specific structure. Appendix E consists of a derivation of the dynamic programming analogue of (5.7), which G_2 applies to arrive at $U_i{}^*$, the sequence $\underline{u}_i{}^*(1)$, $\underline{u}_i{}^*(2)$, ..., $\underline{u}_i{}^*(T)$. This is transmitted to G_{1i}. Then, the iterative procedure progresses as before except that G_{11} and G_{12} use the dynamic programming solution of Appendix C instead of the numerically equivalent (4.41). Now, since G_2 was unable to compute $||II||$ for the particular structure selected, it must attempt to determine it from the state variable sequences. After five control periods have

elapsed, G_2 can determine $||(\underline{X})_1-(\underline{X})_3||$ and $||(\underline{X})_3-(\underline{X})_5||$ by (3.9). Because of Theorem 6.5.1, assuming G_2 is aware that the mapping induced by the control procedure is of the type (6.10), G_2 need only compare $||(\underline{X})_1-(\underline{X})_3||$ with $||(\underline{X})_3-(\underline{X})_5||$ to determine whether or not the chosen structure is "feasible", i.e. has an associated $||\Pi||$ less than unity. Moreover, if G_2 repeats this process for every structure, it can determine the "optimal" structure as the one for which

$$\frac{||(\underline{X})_3-(\underline{X})_5||}{||(\underline{X})_1-(\underline{X})_3||} \tag{6.11}$$

is a minimum.

From the state-transition viewpoint, the self-organizational activity of the 2L3G system is more like our intuition tells us it should be. G_2 selects a structure, the process begins, and if (6.11) is greater than unity, G_2 effects a "re-organization" and the control procedure starts over from scratch. If (6.11) is less than unity, G_2 can be assured that (6.6) will converge to $(\underline{X})_o$, although G_2 may try other structures in an attempt to speed up this convergence.

It is important enough to note again that we have assumed that G_2 is aware that a linear mapping of the type (6.10) is induced by the control procedure under consideration. Without this knowledge, no conclusion could be drawn

from the numerical value of (6.11).

6.6 Applications in Organization Theory

The general problem to which "classical" organization theory addresses itself is the following:[22]

> Given a general purpose for an organization, we can identify the unit tasks necessary to achieve that purpose. The problem is to group these tasks into individual jobs, to group the jobs into administrative units, to group the units into larger units, and finally to establish the top level departments--and to make these groupings in such a way as to minimize the total cost of carrying out all the activities.

In order to formulate an abstract model of an organization, then, according to the above statement, there must be some measure of the effectiveness of any organizational design. This amounts to an ordering relation over the structure set. For example, the "assignment problem", so familiar to operations researchers, gives such a measure. In this thesis, we have proposed another such measure and demonstrated its feasibility by applying it to a special case.

Any normative theory of organizational behavior must include some self-organizational capability, since, as we mentioned in Chapter I, reorganization is a common method of attacking organizational inefficiency.[2] The theory of multi-level, multi-goal systems[17,18] is ideally suited to model this type of activity. In a real organization, the

"bosses" impose organizational changes on their subordinates. In mLnG systems theory, this is represented as we have done in Assumption 6.4.1, by allowing higher level units the capability of re-allocating tasks among the first level units.

In conclusion, we must agree with Mesarović,[17] that a mLnG system is the closest that general systems theory can come to offering an abstract model of an organization. This is reinforced if Churchman's[8] "characteristic (b)" of an organization,

> at any moment of time the organization is pursuing a set of goals,

is accepted.

CHAPTER VII

SUMMARY AND CONCLUSIONS

7.1 Summary

In this investigation we have been concerned with formulating a particular general allocation problem and solving it for an important special case. The problem concerns the allocation of tasks among several interrelated goal-oriented control units which are collectively concerned with controlling a causal subsystem so as to achieve some over-all purpose. Some of these control units, in order to compensate for their being unaware of a portion of the entire system and exerting an effect on only a portion of it, employ a form of adaptive behavior in arriving at their control actions. It is postulated, in the general case, that the time it takes for this adaptive process to reach an equilibrium state, if at all, is a function of the manner in which the tasks are allocated among the control units, i.e. the "organizational structure" of this collection of units. It is argued that, if this is true, the rate of adaptation is a good measure of the effectiveness of the organizational structure, which yields a method of ranking the members of the structure set. One structure is termed "better" than another if the first exhibits a higher adaptation rate than the second.

With an ordering relation over the set of structures, such as the one proposed in this thesis, it is a simple matter conceptually to select the "best" structure from the set. If we allow one of the control units within the system the capability to select the structure, the entire system, when viewed from the outside, exhibits what we have termed "self-organizational" activity; the system changes its own structure in attempting to increase its rate of adaptation.

One characteristic of real organizations is their ability to reorganize themselves. This is accomplished in the manner indicated above; a sub-unit, usually termed a "manager" or "control group",[2] carries out the allocation and re-allocation of tasks. For this reason, we have argued that mathematical models of organizations must have some self-organizational features.

7.2 Conclusions and Future Research

The latter part of this thesis, Chapters III through VI, is concerned with developing the ideas summarized in the previous section for a linear discrete-dynamic system with a quadratic loss function. We conclude, therefore, that these results are applicable in this case. This part of the thesis also demonstrates that it is possible to construct mathematical self-organizing systems which display

a number of conceptual similarities to the intuitive notion of an organization.

The model of organizational behavior developed in this thesis might more properly be termed a "simulation." We have tried to imitate some intuitive conceptions of organizational behavior rather than predict how a "real-world" organization would operate.* In particular, the idea that organizational stability depends on organizational structure, as qualitatively argued by Dubin,[12] is nicely represented in our model.

The idea of applying feedback control theory or "systems analysis" in the study of organizations[20] is not new. However, most of the models involved in these studies are 1L1G systems, so that there is a lack of an appropriate description of the structure. In the theory of mLnG systems, however, any organizational structure could be modelled.

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*The reader may object to our representing groups of humans by "goal-seeking elements." However, at least one eminent management scientist[15] warns that complete rejection by organization theorists of the "mechanical models" of humans precludes the application of many of the recent developments in the information and communication sciences to organization theory.

The simple 2L3G system we have studied illustrates the idea of "influence" or "indirect control", i.e. where the higher level unit adjusts parameters in the performance function of units below it. This idea has also been investigated by Ackoff[1] from the viewpoint of decision theory.

Investigations into mLnG systems with m>2 would be valuable. In particular, this introduces "middle management" goal-seeking elements which indirectly control the causal subsystem while, at the same time, being subjected to control from above. Any self-organizational activity by the top-level unit would then create an entirely new system from the mid-level units' viewpoint. The latter would then be forced to embark upon a self-organizing program of its own, and so on down the line. Models such as these are quite complex, and will require a computer with a large memory to calculate the many contraction factors at the different levels. In addition, "up-and-down" mappings (mentioned briefly in section 2.4) might need to be introduced for mLnG systems with m>2, since the mid-level units will not be able to determine $(\bar{W}_i)_e$ by a procedure such as the one described in section 5.2. For, these units will normally not have complete knowledge of the system.

It would be interesting to study some special cases involving non-linear causal subsystems. Reticulation of multi-variable non-linear systems will most certainly have to be studied numerically. Convergence to equilibrium in the manner that we have used it would probably occur only in certain regions, if at all.

Needless to say, we have only scratched the surface of the theory of multi-level, multi-goal and self-organizing systems.

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APPENDIX A

EXPRESSION OF $\underline{x}(k)$ IN TERMS OF THE INITIAL STATES AND SUBSEQUENT CONTROL ACTIONS

Given the system S obeying

$$\underline{x}(t+1) = A\underline{x}(t)+\underline{z}(t+1)+\underline{m}(t+1), \quad t=0,\ 1,\ \ldots,\ T-1,$$

$$\underline{x}(0)=\underline{c},$$

where $\underline{z}(1)$, $\underline{z}(2)$, ..., $\underline{z}(T)$ is a known sequence of vectors, we can write

$$\underline{x}(1) = A\underline{c}+\underline{z}(1)+\underline{m}(1) \tag{A.1}$$

$$\begin{aligned} \underline{x}(2) &= A\underline{x}(1)+\underline{z}(2)+\underline{m}(2) \\ &= A^2\underline{c}+A\left[\underline{z}(1)+\underline{m}(1)\right]+\underline{z}(2)+\underline{m}(2) \end{aligned}$$

Assume

$$\underline{x}(k) = A^k\underline{c}+\sum_{j=1}^{k} A^{k-j}\left[\underline{z}(j)+\underline{m}(j)\right] \tag{A.2}$$

Then

$$\begin{aligned} \underline{x}(k+1) &= A\underline{x}(k)+\underline{z}(k+1)+\underline{m}(k+1) \\ &= A^{k+1}\underline{c}+A\sum_{j=1}^{k} A^{k-j}\left[\underline{z}(j)+\underline{m}(j)\right]+\underline{z}(k+1)+\underline{m}(k+1) \\ &= A^{k+1}\underline{c}+\sum_{j=1}^{k+1} A^{k+1-j}\left[\underline{z}(j)+\underline{m}(j)\right] \ ; \end{aligned}$$

hence, (A.2) holds by induction (it is true for k=1 from (A.1)).

APPENDIX B

DIFFERENTIATION OF QUADRATIC AND LINEAR FORMS WITH RESPECT TO VECTORS

Given the quadratic form

$$g(\underline{m}) = \underline{m}'A\underline{m} = \sum_{i,j=1}^{n} a_{ij}m_i m_j,$$

where

$$\underline{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix},$$

and A is a symmetric matrix, we make the following definition:

$$\frac{dg}{d\underline{m}} = \begin{bmatrix} \frac{\partial g}{\partial m_1} \\ \frac{\partial g}{\partial m_2} \\ \vdots \\ \frac{\partial g}{\partial m_n} \end{bmatrix};$$

thus,

$$\frac{dg}{d\underline{m}} = \begin{bmatrix} 2\sum_{j=1}^{n} a_{1j}m_j \\ 2\sum_{j=1}^{n} a_{2j}m_j \\ \vdots \\ 2\sum_{j=1}^{n} a_{nj}m_j \end{bmatrix} = 2A\underline{m}.$$

Also, for $f(\underline{m})=\underline{c}'\underline{m}= \sum_{i=1}^{n} c_i m_i$, the same definition yields

$$\frac{df}{d\underline{m}} = \begin{bmatrix} \frac{\partial f}{\partial m_1} \\ \vdots \\ \frac{\partial f}{\partial m_n} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \underline{c}$$

For more complicated forms of the type,

$$g(\underline{m}) = (A\underline{m}+\underline{c})'(A\underline{m}+\underline{c})$$

$$= \underline{m}'A'A\underline{m}+\underline{c}'A\underline{m}+\underline{m}'A'\underline{c}+\underline{c}'\underline{c}$$

we notice that $\underline{c}'A\underline{m} = \underline{m}'A'\underline{c}$, so

$$g(\underline{m}) = \underline{m}'A'A\underline{m}+2\underline{c}'A\underline{m}+\underline{c}'\underline{c} \tag{B.1}$$

and, using the above derivations,

$$\frac{\partial g}{\partial \underline{m}} = 2A'A\underline{m}+2A'\underline{c} = 2A'(A\underline{m}+\underline{c}). \tag{B.2}$$

APPENDIX C

THE DYNAMIC PROGRAMMING SOLUTION OF THE CONTROL PROBLEM OF CHAPTER III

In Chapter III we encounter the following problem: "minimize

$$g(M,X) = \sum_{t=1}^{T} [\underline{x}(t)-\underline{u}(t)]'[\underline{x}(t)-\underline{u}(t)] + \sum_{t=1}^{T} \underline{m}'(t)D\underline{m}(t), \tag{C.1}$$

subject to the constraint

$$\underline{x}(t+1) = A\underline{x}(t)+\underline{m}(t+1)+\underline{z}(t+1), \tag{C.2}$$

for t=0, 1, ..., T-1, with initial conditions $\underline{x}(0)=\underline{c}$."
The vector sequences $\underline{u}(1)$, $\underline{u}(2)$, ..., $\underline{u}(T)$ and $\underline{z}(1)$, $\underline{z}(2)$ ..., $\underline{z}(T)$ are deterministic, and the matrix D is positive definite and diagonal.

Because of the Markovian property of the system (C.2) and the loss function (C.1) we can apply Bellman's principle of optimality to get

$$F(N+1,\underline{c}) = \min_{\underline{m}} \Big\{ [\underline{c}-\underline{u}(T-N-1)]'[\underline{c}-\underline{u}(T-N-1)] + \underline{m}'D\underline{m}+F[N,A\underline{c}+\underline{m}+\underline{z}(T-N)]\Big\} , \tag{C.3}$$

where F(N,$\underline{c}$) denotes the minimum possible contribution to the loss function if the system is in state $\underline{c}$ and N decisions or selections of $\underline{m}(t)$ remain in the process.

Assume F(N,$\underline{c}$) can be expressed in the following form (note ' denotes transposition):

$$F(N,\underline{c}) = \underline{c}'R(N)\underline{c}-2\underline{P}'(N)\underline{c}+V(N) \qquad (C.4)$$

where R(N) is a positive definite matrix, $\underline{P}(N)$ is a vector, and V(N) is a scalar, in particular,

$$F(0,\underline{c}) = \left[\underline{c}-\underline{u}(T)\right]'\left[\underline{c}-\underline{u}(T)\right]$$

$$= \underline{c}'\underline{c}-2\underline{u}'(T)\underline{c}+\underline{u}'(T)\underline{u}(T) \qquad (C.5)$$

so that

$$R(0)=I, \quad \underline{P}(0)=\underline{u}(T), \quad V(0)=\underline{u}'(T)\underline{u}(T) \ . \qquad (C.6)$$

From (C.3) and (C.4),

$$F(N+1,\underline{c}) = \min_{\underline{m}}\Big\{ \underline{c}'\underline{c}-2\underline{u}'(T-N-1)\underline{c}+\underline{u}'(T-N-1)\underline{u}(T-N-1)$$

$$+\underline{m}'D\underline{m}+\left[A\underline{c}+\underline{z}(T-N)+\underline{m}\right]'R(N)\left[A\underline{c}+\underline{z}(T-N)+\underline{m}\right]$$

$$-2\underline{P}'(N)\left[A\underline{c}+\underline{m}+\underline{z}(T-N)\right]+V(N)\Big\} \ . \qquad (C.7)$$

To find the stationary point of the expression in $\{\}$ differentiate it and set it equal to $\underline{0}$:

$$2D\underline{m}+2R(N)\underline{m}+2R(N)\left[A\underline{c}+\underline{z}(T-N)\right]-2\underline{P}(N) = \underline{0} \ ,$$

so

$$\underline{m} = \left[D+R(N)\right]^{-1}\left\{\underline{P}(N)-R(N)\left[A\underline{c}+\underline{z}(T-N)\right]\right\} \ . \qquad (C.8)$$

Substitution of $\underline{m}$ as given by (C.8) yields, after a few algebraic manipulations,

$$F(N+1,\underline{c}) = \underline{c}'\Big\{I+A'\left[R(N)-R(N)\left[D+R(N)\right]^{-1}R(N)\right]A\Big\}\underline{c}$$

$$-2\Big\{\underline{u}(T-N-1)+A'\left[I-R(N)\left[D+R(N)\right]^{-1}\right]\left[\underline{P}(N)-R(N)\underline{z}(T-N)\right]\Big\}'\underline{c}$$

$$+\underline{u}'(T-N-1)\underline{u}(T-N-1)-\underline{P}'(N)\left[D+R(N)\right]^{-1}\underline{P}(N)$$

$$-2\underline{z}'(T-N)\Big\{I-R(N)\left[D+R(N)\right]^{-1}\Big\}\underline{P}(N)$$

$$+\underline{z}'(T-N)\Big\{I-R(N)\left[D+R(N)\right]^{-1}\Big\}R(N)\underline{z}(T-N)+V(N) \qquad (C.9)$$

Now, repeating the assumption made to get (C.4) for N+1 instead of N,

$$F(N+1,\underline{c}) = \underline{c}'R(N+1)\underline{c}-2\underline{P}'(N+1)\underline{c}+V(N+1) \quad . \qquad (C.10)$$

Comparison of (C.10) and (C.9) yields the recursion relations

$$R(N+1) = I+A'\left\{I-R(N)\left[D+R(N)\right]^{-1}\right\}R(N)A \qquad (C.11)$$

$$\underline{P}(N+1) = \underline{u}(T-N-1)+A'\left\{I-R(N)\left[D+R(N)\right]^{-1}\right\}\left[\underline{P}(N)-R(N)\underline{z}(T-N)\right] \qquad (C.12)$$

$$V(N+1) = \underline{u}'(T-N-1)\underline{u}(T-N-1)-\underline{P}'(N)\left[D+R(N)\right]^{-1}\underline{P}(N)$$

$$-2\underline{z}(T-N)\left\{I-R(N)\left[D+R(N)\right]^{-1}\right\}\underline{P}(N)+\underline{z}'(T-N)$$

$$\left\{I-R(N)\left[D+R(N)\right]^{-1}\right\}R(N)\underline{z}(T-N)+V(N) \qquad (C.13)$$

The method of obtaining the optimal policy is:

(1) from (C.6), $R(0)=I$, $\underline{P}(0)=\underline{u}(T)$.

(2) calculate $R(N)$ and $P(N)$ for $N=1, 2, \ldots, T-1$ using the starting values obtained in (1) and the recursion relations (C.11) and (C.12).

(3) From (C.8), we see the optimal policy is, for $N=T-1, T-2, \ldots, 0$,

$$\underline{m}(T-N)=\left[D+R(N)\right]^{-1}\left\{\underline{P}(N)-R(N)\left[A\underline{x}(T-N-1)+\underline{z}(T-N)\right]\right\},$$

since (C.8) is good for any value of the state vector, in particular, the one resulting from applying optimal control up to that point.

The minimum value of the loss function can be found by carrying along V(N) in the recursive procedure in step (2) above. This is simply

$$
\begin{aligned}
g_{min} &= \underline{m}'(1)D\underline{m}(1)+F\left[T-1,\ A\underline{c}+\underline{z}(1)+\underline{m}(1)\right] \\
&= \underline{m}'(1)D\underline{m}(1) \\
&\quad +\left[A\underline{c}+\underline{z}(1)+\underline{m}(1)\right]'R(T-1)\left[A\underline{c}+\underline{z}(1)+\underline{m}(1)\right] \\
&\quad -2\underline{P}'(T-1)\left[A\underline{c}+\underline{z}(1)+\underline{m}(1)\right]+V(T-1) \qquad \text{(C.11)}
\end{aligned}
$$

The above arguments are similar to those used by Adorno.[4]

APPENDIX D

EFFECT OF VARIATION OF D_{11}^* AND D_{12}^* ON $||\Pi_i||$

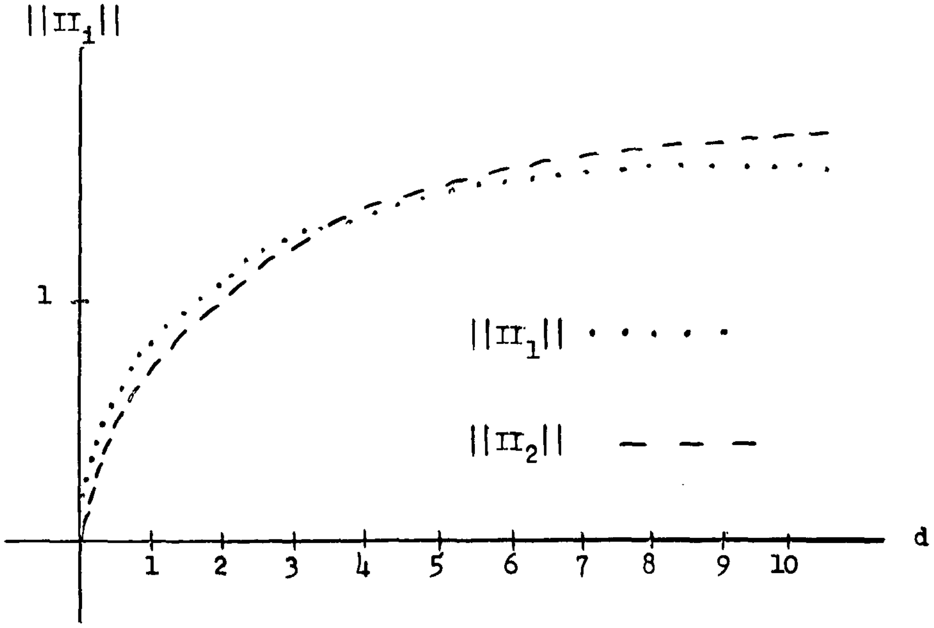
Let A be given by (6.7), and let

$$D_{1i}^* = dI$$

for i=1, 2, and T=3. Figure D-1 shows the results of evaluating $||\Pi_1||$ and $||\Pi_2||$ as a function of d, when

$$\underline{x}_1(t) = \begin{bmatrix} x_3(t) \\ x_5(t) \end{bmatrix} \qquad \underline{x}_2(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} .$$

Figure D-1.



APPENDIX E

THE DYNAMIC PROGRAMMING EQUIVALENT OF (5.7)

From (C.8), in order for G_{1i} to choose its portion of the optimal over-all policy, we must have

$$\left[\left[D+R(N)\right]^{-1}\left\{\underline{P}(N)-R(N)\left[A\underline{c}+\underline{z}(T-N)\right]\right\}\right]_i$$

$$= \left[D_{1i}{}^{*}+R_i(N)\right]^{-1}\left\{\underline{P}_i(N)-R_i(N)\left[A\underline{c}_i+\underline{w}_i(T-N)\right]\right\} \qquad (E.1)$$

where, in the left hand term, the subscript i denotes that these are the elements of the optimal controlled input vector corresponding to the state variables under G_{1i}'s cognizance. G_2 has determined $(W_i)_e$, i.e. $\underline{w}_i{}^e(1)$, $\underline{w}_i{}^e(2)$, ..., $\underline{w}_i{}^e(T)$ in the manner indicated in section 5.2;, thus, from (E.1) we can solve for $\underline{P}_i(N)$, and obtain a formula for $\underline{u}_i{}^*(T-N)$ from (C.12), with appropriate subscripts and $\underline{z}(T-N)$ replaced by $\underline{w}_i{}^e(T-N)$.